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ORDER STATISTICS AND THEIR LINEAR COMBINATIONS FOR THE POWER HALF NORMAL DISTRIBUTION

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Abstract. The power half normal distribution is an extension of the traditional half normal distribution. It introduces a power parameter to the half normal distribution for greater flexibility when modelling skewed data and data with varying tail behaviour. This paper looks at the underlying properties of the power half normal distribution, including the moments, the hazard rate, reliability functions and then the order statistics are introduced along with the L moments of the distribution. The methods are then demonstrated on a real dataset.

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1. Introduction

The Power Half Normal (PHN) distribution can be used to model non-negative data, and this has led to its extensive use in reliability and survival analysis as a lifetime model, in environmental analysis and quality control. There are a large number of theoretical properties underpinning this distribution explained by Gómez and Bolfarine [11]. The investigation of order statistics, particularly their moments, is fundamental to characterizing the behaviour and structural properties of data arising from these distributions.

L-moments are alternative measures to conventional moments, offering robustness and efficiency, particularly in the presence of skewness and outliers. The integration of these moments with the Power Half Normal distributions provides powerful tools for parameter estimation and model validation. This paper aims to explore the moments of order statistics for these distributions and their applications, with a focus on L-moments.

The paper is summarized as follows. Section 2 shows The Power Half Normal (PHN) distribution and theoretical properties. In Section 3, we obtained some expression of order statistics for the single moments from PHN distribution with numerical computations. Section 4 is the applications part based on L-moments with the different data sets. Finally, in Section 5, concluding the whole paper and future work.

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2. Power Half Normal (PHN) Distribution

In this section, we present the development of an extended model derived from the half-normal distribution, known as the Power Half-Normal (PHN) distribution. The PHN class represents a specific case within the broader family of distributions proposed by Pescim et al. [16]. Characterized by two parameters, the PHN distribution is particularly effective for modeling positive-valued data observed in reliability and survival analyses, providing a useful alternative to traditional models such as the half-normal, gamma, and Weibull distributions.

A random variable Z follows a Half-Normal (HN) distribution with scale parameter $\sigma > 0$, $Z \sim HN(\sigma)$ if its probability density function (*pdf*) is

$$f(z;\sigma) = \frac{x}{\sigma}\phi(\frac{x}{\sigma}), \quad z > 0. \tag{1}$$

The extension of the Half-Normal (HN) distribution, referred to as the Power Half-Normal (PHN) distribution, is defined by its *pdf* as follows

$$g(x;\sigma,\alpha) = \frac{2\alpha}{\sigma}\phi(\frac{x}{\sigma})(2\Phi(\frac{x}{\sigma})-1)^{\alpha-1}, \quad -\infty < x < +\infty. \tag{2}$$

The corresponding cumulative distribution function (*cdf*) of the power half normal distribution is

$$G(x;\sigma,\alpha) = \left(2\Phi(\frac{x}{\sigma})-1\right)^{\alpha}, \quad -\infty < x < +\infty, \tag{3}$$

where $\sigma > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter. $\phi(\cdot)$ and $\Phi(\cdot)$ are the *pdf* and *cdf* of the standard Normal distribution, i.e., $N(0,1)$, respectively.

Figure 1 and 2, shows the behaviour of *pdf* and *cdf* of the PHN for varying α, shape parameter and σ, scale parameter are fixed respectively.

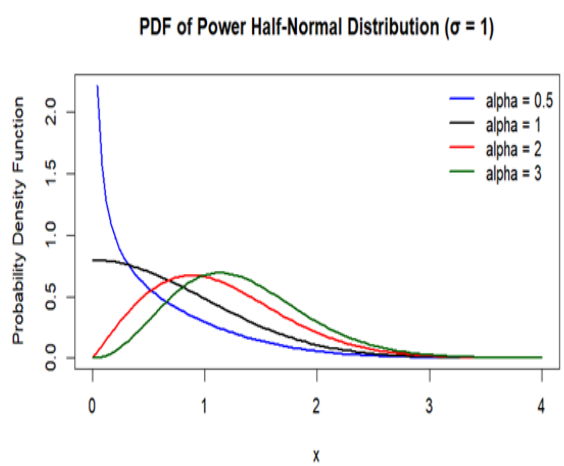

Fig. 1. Behaviour of PDF at different values of parameters

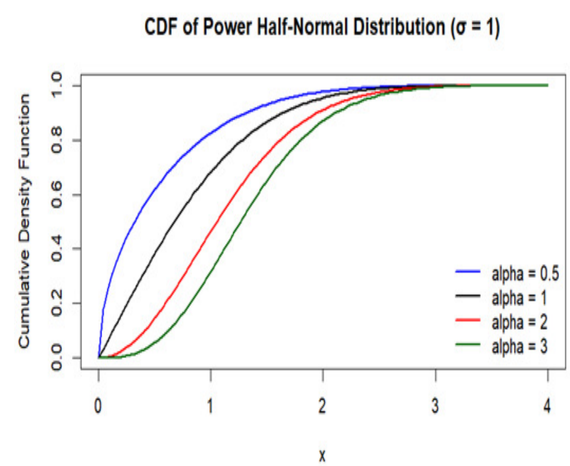

Fig. 2. Behaviour of CDF at different values of parameters

2.1. Mathematical Properties (Gómez and Bolfarine [11]).

If $X \sim PHN(\alpha,\sigma)$ then:

(1) The r-th moment is

$$\mathbb{E}(X^r) = \alpha\sigma^r\kappa_r(\alpha), \quad \text{for} \quad r = 0,1,2,\ldots,$$

where

$$\kappa_r(\alpha) = \int_0^1 \Phi^{-1}\left(\frac{1+u}{2}\right)^r u^{\alpha-1}\,du. \tag{4}$$

(2) Some of the important moments are given as follows:
 (a) $\mathbb{E}(X) = \alpha\sigma\kappa_1$,
 (b) $\mathbb{V}(X) = \alpha\sigma^2(\kappa_2 - \alpha\kappa_1^2)$,
 (c) Skewness Coefficient

$$\sqrt{\beta_1} = \frac{E[(X - E(X))^3]}{(Var(X))^{\frac{3}{2}}} = \frac{\kappa_3 - 3\alpha\kappa_1\kappa_2 + 2\alpha^2\kappa_1^3}{\sqrt{\alpha}(\kappa_2 - \alpha\kappa_1^2)^{3/2}}.$$

 (d) Kurtosis Coefficient

$$\beta_2 = \frac{E[(X - E(X))^4]}{(Var(X))^2} = \frac{\kappa_4 - 4\alpha\kappa_1\kappa_3 + 6\alpha^2\kappa_1^2\kappa_2 - 3\alpha^2\kappa_1^4}{\alpha(\kappa_2 - \alpha\kappa_1^2)^2}.$$

(3) The hazard corresponding hazard $h(x)$ and survivor function $S(x)$ are given as

$$S(x) = 1 - G(x) = 1 - \left(2\Phi\left(\frac{x}{\sigma}\right) - 1\right)^{\alpha},$$

and

$$h(x) = \frac{g(x)}{S(x)} = \frac{\frac{2\alpha}{\sigma}\phi(\frac{x}{\sigma})(2\Phi(\frac{x}{\sigma}) - 1)^{\alpha-1}}{1 - (2\Phi(\frac{x}{\sigma}) - 1)^{\alpha}}.$$

(4) The quantile function

$$Q(p) = \sigma\Phi^{-1}\left(\frac{1 + p^{1/\alpha}}{2}\right), \quad 0 < p < 1.$$

TABLE 1. Statistical Moments with Simulated Values of the PHN Distribution

Parameters		Sim = **1000**				Sim = **10000**			
σ	α	$\mathbb{E}(X)$	$\mathbb{V}(X)$	Skewness	Kurtosis	$\mathbb{E}(X)$	$\mathbb{V}(X)$	Skewness	Kurtosis
0.5	0.5	0.254607	0.077300	1.511439	2.027353	0.256015	0.075946	1.479431	2.261794
	1.0	0.404038	0.098582	0.969086	0.889814	0.401038	0.093434	1.018453	0.953756
	2.0	0.552294	0.084577	0.579355	0.199619	0.563381	0.091246	0.693083	0.337848
	3.0	0.659095	0.082638	0.434473	-0.080741	0.661684	0.085856	0.665009	0.676342
	4.0	0.729232	0.083170	0.645594	0.549127	0.735378	0.081569	0.565259	0.367076
1.0	0.5	0.501307	0.268756	1.405149	1.970267	0.523508	0.307611	1.421679	1.897009
	1.0	0.813090	0.335469	0.866756	0.282925	0.794234	0.358886	0.996969	0.897168
	2.0	1.107562	0.357426	0.673486	0.208243	1.116201	0.357542	0.736579	0.633122
	3.0	1.308383	0.333165	0.622161	0.391215	1.322166	0.340870	0.621121	0.441479
	4.0	1.463383	0.321897	0.640928	0.427213	1.468378	0.332837	0.627693	0.472989
1.5	0.5	0.726125	0.601030	1.509485	2.279009	0.763509	0.682503	1.548490	2.649656
	1.0	1.197424	0.854870	1.030818	0.880071	1.187263	0.793706	0.992945	0.846750
	2.0	1.654105	0.783941	0.627308	0.062790	1.700498	0.814949	0.711112	0.467316
	3.0	2.028926	0.809941	0.643131	0.464955	1.974558	0.778086	0.625695	0.358685
	4.0	2.234382	0.759317	0.457796	-0.213800	2.196897	0.742252	0.601488	0.477852
2.0	0.5	1.046789	1.239050	1.532649	2.467496	1.025177	1.227335	1.480882	2.195900
	1.0	1.648731	1.457832	0.947253	0.667411	1.615574	1.476311	0.980539	0.804640
	2.0	2.245845	1.483437	0.745250	0.432987	2.239292	1.426807	0.694741	0.479830
	3.0	2.620088	1.326578	0.604498	0.281954	2.638914	1.343281	0.568041	0.257951
	4.0	2.932929	1.266651	0.654715	0.586054	2.911444	1.287733	0.607957	0.451862
2.5	0.5	1.289311	1.802503	1.369471	1.904533	1.257123	1.837694	1.513411	2.410173
	1.0	2.017206	2.047275	0.926688	0.787614	1.982241	2.177467	0.988955	0.905961
	2.0	2.797372	2.213858	0.738817	0.762188	2.831248	2.237012	0.691081	0.348129
	3.0	3.323293	2.089888	0.728903	0.831215	3.324349	2.165614	0.604764	0.410100
	4.0	3.600278	2.102822	0.707835	0.496133	3.655176	2.023722	0.552846	0.239453

3. Order Statistics from Power Half Normal Distribution

Order Statistics are a crucial concept in statistics with a wide range of applications in reliability theory, insurance, quality control, insurance and environmental science. They are a fundamental tool in non-parametric statistics and inference and are used in probability theory to analyze random samples from continuous distributions. Order statistics with their applications has been widely described in the well-known books by [7] and [10].

The expression for the moments of order statistics is widely used in statistical literature such as statistical modelling, statistical inferences, decision procedures, nonparametric statistics, among others. Many authors have established the exact expressions and recurrence relations for the several distributions. For example see Alam *et al.* [1–3], Alam and Vidovic [4], Ali and Khan [5], Asgharzadeha *et al.* [8], Nadarajah [14], Genç [6], Nagaraja [15], MirMostafaee [13]and references therein.

Let the random sample $X_1, X_2, ..., X_n$ of size n is obtained from a Power Half Normal distribution with pdf and CDF as in (2) and (3) respectively. If the random sample is arranged in descending order then we have $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ being the order statistics from the power half normal distribution. The pdf of the r-th order statistic $X_{r:n}$ is given by (see, Balakrishnan and Cohen [9]):

$$f_{r:n}(x) = C_{r:n}(F(x))^{r-1}(1-F(x))^{n-r}f(x), \quad 0 \leq r \leq n,\ x > 0, \tag{5}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!},$$

and the k-th moment of $X_{r:n}$ denoted $\mu^k_{r:n}$ is given as

$$\mu^{(k)}_{r:n} = \int_0^\infty x^k f_{r:n}(x)dx, \tag{6}$$

from (2), (5) and (6), we have

$$\mu^{(k)}_{r:n} = \int_0^\infty x^k \frac{2\alpha}{\sigma}\phi\left(\frac{x}{\sigma}\right)\left(2\Phi\left(\frac{x}{\sigma}\right)-1\right)^{\alpha r-1}\left(1-\left(2\Phi\left(\frac{x}{\sigma}\right)-1\right)^{\alpha}\right)^{n-r}. \tag{7}$$

Letting u $= 2\Phi(x/\sigma) - 1$ and using the binomial expansion, we obtain

$$\mu^{(k)}_{r:n} = C_{r:n}\int_0^1 \sigma^k(\Phi^{-1}(\frac{u+1}{2}))^k u^{\alpha r-1}\sum_{l=0}^{n-r}\binom{n-r}{l}u^{\alpha l}(-1)^l du.$$

Simplifying

$$\begin{aligned}\mu^{(k)}_{r:n} &= \alpha\sigma^k C_{r:n}\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}\int_0^1(\Phi^{-1}(\frac{u+1}{2}))^k u^{\alpha r-\alpha l-1}du\\ &= \alpha\sigma^k C_{r:n}\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}\kappa_k(\alpha(r-l)),\end{aligned}$$

where

$$\kappa_k(\gamma) = \int_0^1(\Phi^{-1}(\frac{x+1}{2}))^k x^{\gamma-1}dx, \quad 0 < x < 1. \tag{8}$$

Remark 3.1. *When $n = r = 1$ in (8), we get*

$$\mu^{(k)}_{1:1} = \mathbb{E}(X^k) = \alpha\sigma^k\kappa_k(\alpha), \tag{9}$$

which is the k-th moments of PHN distribution (Gómez and Bolfarine [11]).

At $k = 1, 2, 3$ and $k = 4$ in (9), we obtain the first four moments of the PHN distribution as

$$\mu_{1:1}^{(1)} = \mathbb{E}(X) = \alpha\sigma\kappa(\alpha), \tag{10}$$

and

$$\mu_{1:1}^{(2)} = \mathbb{E}(X^2) = \alpha\sigma^2\kappa_2(\alpha), \tag{11}$$

which upon using (10) and (11) together with $\mathbb{V}[X] = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$, gives the variance of PHN distribution as

$$\mathbb{V}[X] = \alpha\sigma\left[\sigma\kappa_2(\alpha) - \kappa(\alpha)\right]. \tag{12}$$

Remark 3.2. *At $r = 1$, in (8), we get*

$$\mu_{1:n}^{(k)} = n\,\alpha\sigma^k \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \kappa_k(\alpha(1-l)), \tag{13}$$

and at $r = n$, in (8), we get

$$\mu_{n:n}^{(k)} = n\alpha\sigma^k\kappa_k(\alpha(n)). \tag{14}$$

The expressions in (13) and (14) are the extreme $k-th$ moments of the PHN distribution.

As exact moments are expressed in the term of $\kappa_r(\cdot)$ function, which cannot be calculated analytically because of the complexity of finding the inverse of the normal distribution and the mathematical form of the $\Phi^{-1}(\cdot)$. So, we are using the iterative methods like the inverse *cdf* to simulate the data and to find the moments of the order statistics of PHN distribution.

3.1. **Numerical Computations.** In this section, we present the simulated moments of order statistics for the PHN distribution corresponding to the parameter values $\alpha = 2$ and $\sigma = 1$. The results for sample sizes $n = 1$ to 10 are summarized in Tables 2 and 3. For the order statistics, the identity

$$\sum_{r=1}^{n} \mu_{r:n} = n\,E(X),$$

(see Nagaraja and David [10]) is used to verify the calculations of the means and variances. This relationship holds approximately, with minor deviations arising from the numerical approximation of the integral $\kappa_k(\alpha(r-l))$. The results are generated using simulation with Sim = 1000 and Sim = 10000 iterations to reduce computational error.

TABLE 2. Simulated Moments of Order Statistics for PHN Distribution

Moments and Variance at Sim=1000						
n	r	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$	Var(X)
1	1	1.142245	1.707215	3.071579	6.359578	0.402491
2	1	0.783012	0.773041	0.902563	1.210978	0.159933
	2	1.455012	2.423084	4.515099	9.268064	0.306023
3	1	0.632225	0.510481	0.485583	0.522058	0.110773
	2	1.073521	1.336880	1.890260	2.979461	0.184433
	3	1.639765	2.988697	5.986925	13.097919	0.299868
4	1	0.567119	0.408617	0.347386	0.336236	0.086993
	2	0.909246	0.941488	1.086633	1.379762	0.114759
	3	1.268648	1.770937	2.680714	4.351619	0.161470

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n	r	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$	Var(X)
	4	1.781868	3.459766	7.299063	16.679979	0.284712
5	1	0.500105	0.319923	0.240810	0.204709	0.069818
	2	0.802506	0.731931	0.743007	0.827180	0.087914
	3	1.079395	1.274376	1.626324	2.223488	0.109283
	4	1.402021	2.116414	3.410532	5.825252	0.150752
	5	1.885198	3.815450	8.247404	18.949842	0.261477
6	1	0.456354	0.266074	0.183241	0.142862	0.057815
	2	0.714294	0.579332	0.519941	0.507514	0.069117
	3	0.940967	0.967836	1.075525	1.279222	0.082418
	4	1.205404	1.560587	2.155788	3.157994	0.107589
	5	1.495403	2.384179	4.028274	7.177625	0.147949
	6	1.961375	4.105272	9.145377	21.637099	0.258281
7	1	0.432080	0.237967	0.155031	0.114702	0.051274
	2	0.667448	0.505214	0.423802	0.387422	0.059727
	3	0.880492	0.843726	0.869755	0.954959	0.068459
	4	1.080753	1.243753	1.512302	1.930237	0.075726
	5	1.315178	1.827091	2.674980	4.121562	0.097398
	6	1.597244	2.679989	4.716740	8.701039	0.128801
	7	2.041985	4.398748	9.974350	23.756671	0.229047
8	1	0.393270	0.201018	0.122438	0.084872	0.046357
	2	0.623434	0.445394	0.356292	0.315203	0.056724
	3	0.806624	0.712703	0.681513	0.699199	0.062060
	4	0.989322	1.046283	1.174599	1.392249	0.067525
	5	1.175750	1.460507	1.906362	2.602719	0.078120
	6	1.398764	2.054055	3.158598	5.074190	0.097514
	7	1.665467	2.907101	5.309153	10.127710	0.133320
	8	2.097943	4.640632	10.805372	26.424787	0.239267
9	1	0.383486	0.186206	0.106256	0.068465	0.039145
	2	0.582351	0.382478	0.276322	0.215877	0.043346
	3	0.747436	0.604582	0.523949	0.482999	0.045921
	4	0.906210	0.871133	0.883130	0.939672	0.049917
	5	1.062596	1.186784	1.388902	1.698795	0.057674
	6	1.240430	1.611312	2.187057	3.095463	0.072645
	7	1.433478	2.143942	3.339525	5.407900	0.089084
	8	1.693856	2.994520	5.515625	10.567899	0.125373
	9	2.139076	4.811779	11.389615	28.388222	0.236135
10	1	0.352767	0.157785	0.083329	0.049838	0.033341
	2	0.548505	0.340971	0.234686	0.176168	0.040113
	3	0.703105	0.537689	0.442276	0.388036	0.043333
	4	0.850373	0.769170	0.735431	0.740078	0.046035
	5	0.992056	1.031362	1.119618	1.265487	0.047188
	6	1.146425	1.374562	1.720550	2.245494	0.060272
	7	1.307375	1.779895	2.518501	3.697816	0.070667
	8	1.499054	2.333496	3.766422	6.295328	0.086332

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n	r	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$	Var(X)
	9	1.754903	3.202598	6.068736	11.925549	0.122915
	10	2.169973	4.931787	11.739220	29.255624	0.223005

Table 3. Simulated Moments of Order Statistics for the PHN Distribution

		Moments and Variance at Sim=10000				
n	r	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$	Var(X)
1	1	1.130946	1.636605	2.804799	5.472703	0.357566
2	1	0.795675	0.807802	0.968149	1.314563	0.174704
	2	1.462939	2.455618	4.608910	9.500156	0.315428
3	1	0.649210	0.535910	0.520592	0.572337	0.114435
	2	1.091132	1.356278	1.873365	2.828742	0.165708
	3	1.661114	3.063801	6.194074	13.589970	0.304503
4	1	0.556064	0.393181	0.327335	0.308664	0.083974
	2	0.902616	0.928177	1.060837	1.325110	0.113461
	3	1.267191	1.765066	2.671873	4.356976	0.159293
	4	1.787523	3.470800	7.265127	16.292146	0.275561
5	1	0.496941	0.316568	0.238801	0.204422	0.069618
	2	0.794584	0.718550	0.722988	0.797165	0.087186
	3	1.070056	1.253416	1.588601	2.159427	0.108396
	4	1.392565	2.090246	3.358085	5.743915	0.151008
	5	1.881968	3.809913	8.255795	19.068035	0.268109
6	1	0.457086	0.267481	0.184858	0.144718	0.058554
	2	0.716265	0.583262	0.525830	0.515770	0.070227
	3	0.952651	0.990353	1.110256	1.330231	0.082809
	4	1.195383	1.530992	2.085028	3.001125	0.102052
	5	1.489270	2.357032	3.945994	6.961678	0.139107
	6	1.952651	4.068971	9.027969	21.282415	0.256125
7	1	0.421618	0.227169	0.144810	0.104986	0.049407
	2	0.658730	0.492422	0.407040	0.365832	0.058497
	3	0.863806	0.813394	0.824498	0.891473	0.067234
	4	1.069964	1.222242	1.480446	1.891617	0.077419
	5	1.290052	1.759537	2.526533	3.806031	0.095303
	6	1.568217	2.591574	4.498911	8.180712	0.132270
	7	2.022795	4.330963	9.795765	23.357685	0.239263
8	1	0.394135	0.197451	0.116355	0.077570	0.042109
	2	0.610735	0.422359	0.322888	0.268415	0.049362
	3	0.795391	0.688935	0.642547	0.639742	0.056288
	4	0.971749	1.008553	1.110360	1.289511	0.064256
	5	1.159540	1.420194	1.829621	2.470441	0.075660
	6	1.371429	1.972119	2.965162	4.650663	0.091300

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n	r	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$	Var(X)
	7	1.636578	2.805372	5.027773	9.406966	0.126984
	8	2.076169	4.550668	10.516317	25.587701	0.240191
9	1	0.371120	0.174698	0.096427	0.059906	0.036968
	2	0.574184	0.373127	0.267785	0.208822	0.043439
	3	0.744231	0.602403	0.524125	0.485846	0.048524
	4	0.903074	0.869808	0.887662	0.954730	0.054266
	5	1.062837	1.191698	1.403523	1.729656	0.062074
	6	1.238702	1.608885	2.184572	3.092753	0.074503
	7	1.438379	2.160318	3.380216	5.498806	0.091383
	8	1.699602	3.012745	5.562479	10.684320	0.124097
	9	2.125008	4.740540	11.096652	27.256838	0.224880
10	1	0.352693	0.158500	0.083894	0.050224	0.034107
	2	0.543095	0.333827	0.226817	0.167722	0.038875
	3	0.700737	0.534748	0.439452	0.385592	0.043716
	4	0.844760	0.762057	0.729302	0.736338	0.048437
	5	0.989711	1.034069	1.135577	1.305914	0.054541
	6	1.141350	1.363893	1.700935	2.207664	0.061212
	7	1.304033	1.772308	2.504439	3.671623	0.071805
	8	1.499288	2.336342	3.777113	6.324649	0.088477
	9	1.748769	3.177007	5.987359	11.690795	0.118812
	10	2.169519	4.933518	11.758247	29.368006	0.226705

4. Applications

In this section we will use the method of L-Moments to estimate the parameters of the power half-normal distribution. We will apply it to the Zomato delivery times dataset obtained from kaggle.com and we will also use the flood levels dataset to compare with the Normal and exponential distribution. The L-Moments are more resilient to outliers than the traditional moments, they also exist for heavy tailed distributions like the Cauchy, as they do not depend on the higher powers of the data like traditional moments. Thus they can be used in Extreme Value distribution estimations, parameter estimation, hypothesis testing and many other applications.

The L-Moments according to Hosking [12] are expectations of particular linear combinations of expectations of Order statistics. We will apply the L-Moments to estimate the parameters in this section to the normal and exponential distribution to the data and after the goodness of fit will be checked on the basis of the Kolmogorov Smirnov goodness of fit test is used to evaluate and compare the different distributions. The higher the p-value of the K-S test the better the fit of the distribution.

4.1. L-Moments.

Consider a random sample $X_1, X_2, \ldots, X_n$ of size n, drawn from a Power Half Normal distribution with the probability density function described in (2). Letting $X_{i:n}$ be the i-th order statistic for the sample where $i = 1, 2, \ldots, n$, the L-Moments [12] are a linear combinations of the Order Statistics. The L-Moments are calculated from the following formula:

$$L_r = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\mu_{r-j:j}\ ,\ r \geq 1, \tag{15}$$

where $\mu_{r:n}$ is as defined in (6) for $k = 1$.
For $r = 1, 2$ in (15), the first and second L-Moments are given as:

$$L_1 = E(X_{1:1}) \text{ and } L_2 = \frac{1}{2}E(X_{2:2} - X_{1:2}) \tag{16}$$

The first and second sample L-Moments [12] are calculated as:

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} x_{i:n} \text{ and } l_2 = \frac{2}{n(n-1)}\sum_{i=2}^{n}(i-1)x_{i:n} - l_1 \tag{17}$$

Equating $L_1 = l_1$ and $L_2 = l_2$ in (16) and (17), we obtain the L-moment estimator $\hat{\alpha}$ and $\hat{\sigma}$ for α and σ respectively. Now applying this to the power half normal distribution we have:

$$l_1 = \alpha\sigma\kappa_1(\alpha) \tag{18}$$

$$l_2 = \alpha\sigma(\kappa_1(2\alpha) - \kappa_1(\alpha)) \tag{19}$$

Equations (18) and (19) cannot be solved analytically and we will rely on numerical techniques to obtain the estimates $\hat{\alpha}$ and $\hat{\sigma}$, so we are solving the following equation by the software Python.

4.2. **Examples. Data-1: Flood levels data:** The data represents the maximum flood levels (in millions of cubic feet per second) of the Susquehanna River at Harrisburg, Pennsylvania over 20 four-year periods (1890-1969) as: 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265. From Table (4) and Table (5), the flood data is leptokurtic and fits to the PHN distribution respectively.

Data-2: Zomato Delivery Time Data: The data consist of the delivery time in minutes for Zomato orders from different cities over a one year period. The first 20 values of the data is given as: 23, 21, 20, 41, 20, 33, 40, 41, 15, 36, 26, 20, 39, 34, 15, 18, 38, 47, 12, 26 and data is assumed to be independent. Table (4) shows a Mean of 28.25 and Median of 26.00, indicating a slightly higher average and a mild right skew. The Standard Deviation of 10.38 suggests moderate spread within the 12–47 range. With Skewness of 0.14, the distribution is nearly symmetric, and the Kurtosis of –1.35 indicates a flatter, platykurtic shape with lighter tails than a normal distribution.

Table 4. Table showing the summary of the datasets

Data	**Min.**	**Median**	**Mean**	**Variance**	**Max.**	**Skewness**	**Kurtosis**
Flood data	0.265	0.407	0.423	0.0157	0.740	1.068	3.600
Zomato data	12.00	26.00	28.25	107.74	47.00	0.14	-1.35

4.3. **Assessment of Model Adequacy.** The results of this study for the flood level data and Zomato delivery time data sets are shown in Table (5) and Table (6) and the Power half normal shows overall better performance as compare to Normal Distribution and Exponential Distribution.

- Normal distribution

$$g(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}; \quad -\infty < x < +\infty, -\infty < \mu < +\infty, \sigma^2 > 0.$$

- Exponential distribution

$$g(x; \mu, \sigma) = \frac{1}{\sigma} \exp\left(-\frac{x-\mu}{\sigma}\right); \quad x \geq \mu, \sigma > 0.$$

TABLE 5. Table showing the goodness of fit for the Flood Levels dataset-1

Model	Estimates	KS Test Statistic	P-value
Power Half Normal Distribution	$\alpha = 18.01$, $\sigma = 0.19$	0.127666	0.860062
Normal Distribution	$\mu = 0.42$, $\sigma = 0.12$	0.200490	0.349868
Exponential Distribution	$\mu = 0.26$, $\sigma = 0.16$	0.213654	0.278931

TABLE 6. Table showing the goodness of fit for the Zomato Delivery Time dataset-2

Model	Estimates	KS Test Statistic	P-value
Power Half Normal Distribution	$\alpha = 3.19$, $\sigma = 20.60$	0.141292	0.768733
Normal Distribution	$\mu = 28.25$, $\sigma = 10.38$	0.157511	0.647474
Exponential Distribution	$\mu = 12.00$, $\sigma = 16.25$	0.188786	0.421978

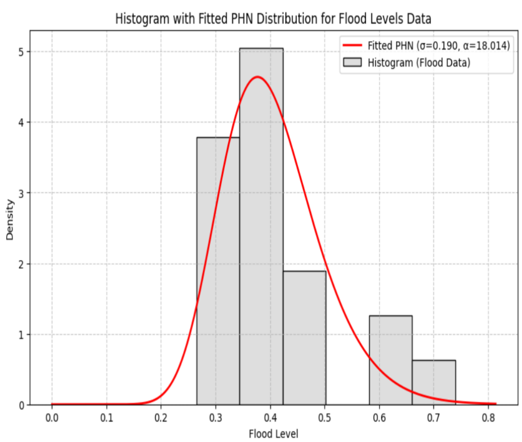

Fig. 3. Estimated *pdf* of the considered model for the first data set-1

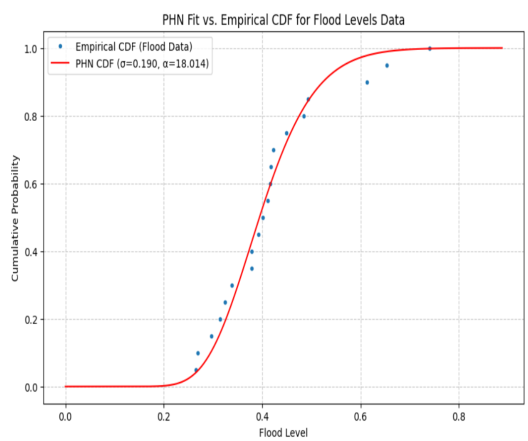

Fig. 4. Estimated *cdf* of the considered models for the first data set-1

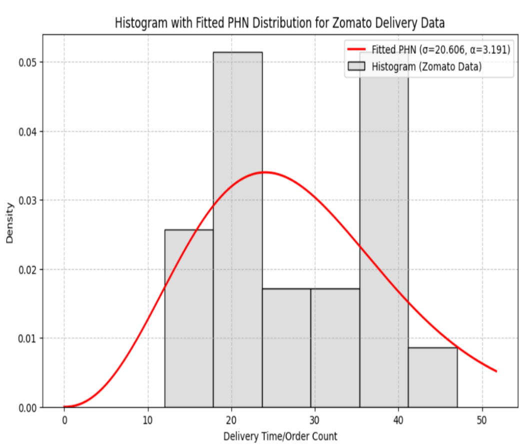

Fig. 5. Estimated *pdf* of the considered model for the first data set-2

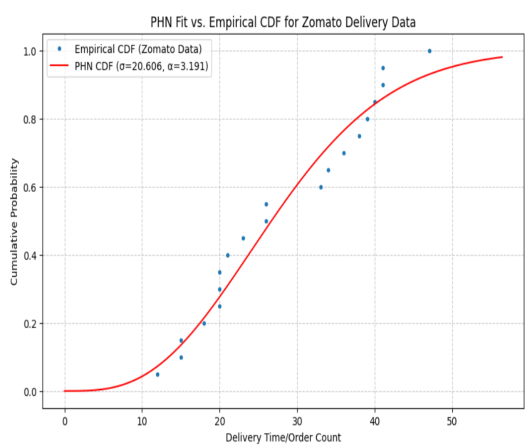

Fig. 6. Estimated *cdf* of the considered models for the first data set-2

5. Conclusion and Future work

In this study, we highlight the important significance of order statistics (OSs) in many theoretical and practical applications, including goodness-of-fit evaluations and L-moments estimation. This work is mainly concerned with the extraction of L-moments of order statistics from the PHN Distribution. When it comes to solving problems involving moments of OSs, the PHN displays more versatility than more conventional distributions and outperforms the normal distribution as it is able to model non-negative data which is more common in day to data examples of events. Two real datasets were used to apply the Power half Normal distribution results, which addressed parameter estimation issues and gauged the goodness of fit. The findings show that the Power Half Normal distribution is a useful tool for both theoretical and practical applications since it can manage the complexity related to moments of OSs in an efficient manner. These findings can be further extended to more advanced frameworks such as generalized order statistics and progressive censoring schemes. Such extensions offer promising directions for future research.

Declaration

The authors declare no competing interests. We consent to the publication of the data and materials. This study did not generate or utilize any new datasets; therefore, data sharing is not applicable. We are using the software Python to calculate the numerical values and graphics. The authors only used Grammarly and QuillBot to improve their grammar and spelling. They are solely responsible for the content of the manuscript and all thoughts and analyses are their own.

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