



p -SCHATTEN NORM INEQUALITIES FOR ČEBYŠEV'S FUNCTIONAL VIA A BIERNACKI, PIDEK AND RYLL-NARDZEWSKI RESULT

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ABSTRACT. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\text{tr}(|A|^p)]^{1/p} < \infty.$$

For $\mathbf{A} = (A_1, \dots, A_n)$, $\mathbf{B} = (B_1, \dots, B_n) \in \mathcal{B}(H)$ we define the noncommutative operator Čebyšev functional

$$D_n(\mathbf{A}, \mathbf{B}) := \frac{1}{n} \sum_{i=1}^n A_i B_i - \frac{1}{n} \sum_{i=1}^n A_i \frac{1}{n} \sum_{i=1}^n B_i.$$

and we provide several bounds for the p -Schatten norm $\|D_n(\mathbf{A}, \mathbf{B})\|_p$ with $p \geq 1$ in terms of forward difference defined by $\Delta A_k := A_{k+1} - A_k$, $1 \leq k \leq n-1$, $n \geq 2$.

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1. INTRODUCTION

In 1935, G. Grüss [10] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integral means integrals as follows:

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \quad (1.1)$$

$$\leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma)$$

where $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable on $[a, b]$ and satisfying the assumption

$$\varphi \leq f(x) \leq \Phi, \quad \gamma \leq g(x) \leq \Gamma$$

for each $x \in [a, b]$ where $\varphi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss' type see the Chapter X of the recent book [11] by Mitrinović, Pečarić and Fink.

Let $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$ be two n -tuples of real numbers. We consider the Čebyšev's functional

$$D_n(\mathbf{a}, \mathbf{b}) := \frac{1}{n} \sum_{i=1}^n a_i b_i - \frac{1}{n} \sum_{i=1}^n a_i \frac{1}{n} \sum_{i=1}^n b_i, \quad n \geq 2. \quad (1.2)$$

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [3] established the following discrete version of Grüss' inequality, see also [11, Ch. X]:

Theorem 1.1. *Let $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$ be two n -tuples of real numbers such that $r \leq a_i \leq R$ and $s \leq b_i \leq S$ for $i = 1, \dots, n$. Then one has the inequality:*

$$\begin{aligned} |D_n(\mathbf{a}, \mathbf{b})| &\leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (R - r) (S - s) \\ &\leq \frac{1}{4} (R - r) (S - s), \end{aligned} \quad (1.3)$$

when $[x]$ is the integer part of x , $x \in \mathbb{R}$.

In 1981, A. Lupaş [11, Ch. X] proved some similar results for the first difference of $\mathbf{a}$ as follows :

Theorem 1.2. *Let $\mathbf{a}, \mathbf{b}$ be two monotonic n -tuples in the same sense and $\mathbf{p}$ a positive n -tuple. Then*

$$\begin{aligned} 0 &\leq \min_{1 \leq i \leq n-1} |a_{i+1} - a_i| \min_{1 \leq i \leq n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n} \sum_{i=1}^n i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^n i p_i \right)^2 \right] \\ &\leq \frac{1}{P_n} \sum_{i=1}^n p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^n p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^n p_i b_i \\ &\leq \max_{1 \leq i \leq n-1} |a_{i+1} - a_i| \max_{1 \leq i \leq n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n} \sum_{i=1}^n i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^n i p_i \right)^2 \right]. \end{aligned} \quad (1.4)$$

If there exists the numbers $\bar{a}, \bar{a}_1, r, r_1, (rr_1 > 0)$ such that $a_k = \bar{a} + kr$ and $b_k = \bar{a}_1 + kr_1$, then in (1.4) the equality holds.

In particular, if $p_i = \frac{1}{n}$, then by (1.4) we derive

$$\begin{aligned} 0 &\leq \frac{1}{12} (n^2 - 1) \min_{1 \leq i \leq n-1} |a_{i+1} - a_i| \min_{1 \leq i \leq n-1} |b_{i+1} - b_i| \\ &\leq D_n(\mathbf{a}, \mathbf{b}) \\ &\leq \frac{1}{12} (n^2 - 1) \max_{1 \leq i \leq n-1} |a_{i+1} - a_i| \max_{1 \leq i \leq n-1} |b_{i+1} - b_i|, \end{aligned} \quad (1.5)$$

for $\mathbf{a}, \mathbf{b}$ two monotonic n -tuples in the same sense.

For a bounded linear operator A in the complex Hilbert space H we define the operator modulus $|A| := (A^*A)^{1/2}$.

In order to extend these results for p -Schatten norms we need the following preparations.

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H . If $\{e_i\}_{i \in I}$ an orthonormal basis of H , we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \quad (1.6)$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\text{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle, \quad (1.7)$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H . Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.7) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1.3. *We have:*

(i) *If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and*

$$\text{tr}(A^*) = \overline{\text{tr}(A)}; \quad (1.8)$$

(ii) *If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and*

$$\text{tr}(AT) = \text{tr}(TA) \text{ and } |\text{tr}(AT)| \leq \|A\|_1 \|T\|; \quad (1.9)$$

(iii) *$\text{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\text{tr}\| = 1$;*

(iv) *If $A, B \in \mathcal{B}_2(H)$ then $AB, BA \in \mathcal{B}_1(H)$ and $\text{tr}(AB) = \text{tr}(BA)$;*

(v) *$\mathcal{B}_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_1(H)$.*

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the p -Schatten norm is finite [14, p. 60-64]

$$\|A\|_p := [\text{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \quad (1.10)$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \quad (1.11)$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$ -ideal $\mathcal{B}_p(H)$ and $(\mathcal{B}_p(H), \|\cdot\|_p)$ is a Banach space.

Also, see for instance [14, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \quad A \in \mathcal{B}_p(H) \quad (1.12)$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \quad A, B \in \mathcal{B}_p(H) \quad (1.13)$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \quad \|BA\|_p \leq \|B\| \|A\|_p, \quad A \in \mathcal{B}_p(H), \quad B \in \mathcal{B}(H). \quad (1.14)$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \quad A \in \mathcal{B}_p(H), \quad B, C \in \mathcal{B}(H). \quad (1.15)$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \quad A \in \mathcal{B}_p(H), \quad B \in \mathcal{B}_q(H). \quad (1.16)$$

For the theory of trace functionals and their applications the reader is referred to [13] and [14].

For some classical trace inequalities see [7], [8], and [12], which are continuations of the work of Bellman [1].

Motivated by the above numerical inequalities that provide upper bounds for the Čebyšev's functional (1.2), in this paper we consider the noncommutative operator version defined by

$$D_n(\mathbf{A}, \mathbf{B}) := \frac{1}{n} \sum_{i=1}^n A_i B_i - \frac{1}{n} \sum_{i=1}^n A_i \frac{1}{n} \sum_{i=1}^n B_i, \quad n \geq 2$$

for $\mathbf{A} = (A_1, \dots, A_n), \mathbf{B} = (B_1, \dots, B_n) \in B(H)$ and provide several bounds for the *p-Schatten norm* $\|D_n(\mathbf{A}, \mathbf{B})\|_p$ with $p \geq 1$ in terms of forward difference defined by $\Delta A_k := A_{k+1} - A_k, 1 \leq k \leq n-1, n \geq 2$.

2. 1-SCHATTEN NORM INEQUALITIES

We can state the following result for the 1-Schatten norm:

Theorem 2.1. *If $\mathbf{A} = (A_1, \dots, A_n) \in (B_p(H))^n$ and $\mathbf{B} = (B_1, \dots, B_n) \in (B_q(H))^n$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} \|D_n(\mathbf{A}, \mathbf{B})\|_1 &\leq D_n(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q}) \\ &\leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q \\ &\leq \frac{1}{4} \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q, \end{aligned} \quad (2.1)$$

where

$$\mathbf{V}_{\mathbf{A},p} := \left(0, \|\Delta A_1\|_p, \dots, \sum_{k=1}^{n-1} \|\Delta A_k\|_p \right) \quad \text{and} \quad \mathbf{V}_{\mathbf{B},q} := \left(0, \|\Delta B_1\|_q, \dots, \sum_{k=1}^{n-1} \|\Delta B_k\|_q \right).$$

In particular, if $\mathbf{A}, \mathbf{B} \in (B_2(H))^n$, then

$$\begin{aligned} \|D_n(\mathbf{A}, \mathbf{B})\|_1 &\leq D_n(\mathbf{V}_{\mathbf{A},2}, \mathbf{V}_{\mathbf{B},2}) \\ &\leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \sum_{k=1}^{n-1} \|\Delta A_k\|_2 \sum_{k=1}^{n-1} \|\Delta B_k\|_2 \\ &\leq \frac{1}{4} \sum_{k=1}^{n-1} \|\Delta A_k\|_2 \sum_{k=1}^{n-1} \|\Delta B_k\|_2. \end{aligned} \quad (2.2)$$

Proof. Let us start with the following identity which can be proved by direct computation for $\mathbf{A} = (A_1, \dots, A_n) \in (B_p(H))^n$ and $\mathbf{B} = (B_1, \dots, B_n) \in (B_q(H))^n$

$$D_n(\mathbf{A}, \mathbf{B}) = \frac{1}{2n^2} \sum_{i,j=1}^n (A_j - A_i)(B_j - B_i) = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (A_j - A_i)(B_j - B_i). \quad (2.3)$$

The proof of the first equality in (2.3) follows by multiplying the brackets and then summing over i and j from 1 to n . The second equality is obvious by symmetry reasons over the indices i and j .

As $i < j$ we can write that $A_j - A_i = \sum_{k=i}^{j-1} \Delta A_k$ and $B_j - B_i = \sum_{k=i}^{j-1} \Delta B_k$. Using the generalized triangle inequality and the property (1.16) we have successively from (2.3) that:

$$\begin{aligned} \|D_n(\mathbf{A}, \mathbf{B})\|_1 & \\ &= \frac{1}{n^2} \left\| \sum_{1 \leq i < j \leq n} \sum_{k=i}^{j-1} \Delta A_k \sum_{k=i}^{j-1} \Delta B_k \right\|_1 \leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \left\| \sum_{k=i}^{j-1} \Delta A_k \sum_{k=i}^{j-1} \Delta B_k \right\|_1 \\ &\leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \left\| \sum_{k=i}^{j-1} \Delta A_k \right\|_p \left\| \sum_{k=i}^{j-1} \Delta B_k \right\|_q \\ &\leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{k=i}^{j-1} \|\Delta B_k\|_q \\ &= \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \left(\sum_{k=1}^{j-1} \|\Delta A_k\|_p - \sum_{k=1}^{i-1} \|\Delta A_k\|_p \right) \left(\sum_{k=1}^{j-1} \|\Delta B_k\|_q - \sum_{k=1}^{i-1} \|\Delta B_k\|_q \right) \\ &= \frac{1}{2n^2} \sum_{i,j=1}^n \left(\sum_{k=1}^{j-1} \|\Delta A_k\|_p - \sum_{k=1}^{i-1} \|\Delta A_k\|_p \right) \left(\sum_{k=1}^{j-1} \|\Delta B_k\|_q - \sum_{k=1}^{i-1} \|\Delta B_k\|_q \right) \\ &= D_n(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q}). \end{aligned} \quad (2.4)$$

Observe that

$$0 \leq \sum_{k=1}^{j-1} \|\Delta A_k\|_p \leq \sum_{k=1}^{n-1} \|\Delta A_k\|_p$$

and

$$0 \leq \sum_{k=1}^{j-1} \|\Delta B_k\|_q \leq \sum_{k=1}^{n-1} \|\Delta B_k\|_q$$

for all $j \in \{1, \dots, n\}$.

Utilising the inequality (1.3) for $\mathbf{V}_\mathbf{A}$ and $\mathbf{V}_\mathbf{B}$ we have

$$\begin{aligned} (0 \leq) D_n(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q}) &\leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q \\ &\leq \frac{1}{4} \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q \end{aligned}$$

and by (2.4) we get (2.1). □

We have for $\mathbf{e} = (1, 2, \dots, n)$ that

$$D_n(\mathbf{a}, \mathbf{e}) := \frac{1}{n} \sum_{i=1}^n i a_i - \frac{n+1}{2} \frac{1}{n} \sum_{i=1}^n a_i$$

and

$$\begin{aligned} D_n(\mathbf{e}, \mathbf{e}) &:= \frac{1}{n} \sum_{i=1}^n i^2 - \frac{n+1}{2} \frac{1}{n} \sum_{i=1}^n i \\ &= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{1}{12} (n^2 - 1). \end{aligned}$$

If we use the *pre-Grüss inequality*

$$|D_n(\mathbf{a}, \mathbf{b})|^2 \leq D_n(\mathbf{a}, \mathbf{a}) D_n(\mathbf{b}, \mathbf{b}),$$

we get

$$|D_n(\mathbf{a}, \mathbf{e})|^2 \leq \frac{1}{12} (n^2 - 1) D_n(\mathbf{a}, \mathbf{a}). \quad (2.5)$$

If $r \leq a_i \leq R$ for $i = 1, \dots, n$, then by (1.3) we have

$$D_n(\mathbf{a}, \mathbf{a}) \leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (R - r)^2 \leq \frac{1}{4} (R - r)^2. \quad (2.6)$$

By (2.5) and (2.6) we also have

$$\begin{aligned} |D_n(\mathbf{a}, \mathbf{e})|^2 &\leq \frac{1}{12} (n^2 - 1) D_n(\mathbf{a}, \mathbf{a}) \\ &\leq \frac{1}{12} (n^2 - 1) \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (R - r)^2 \\ &\leq \frac{1}{48} (n^2 - 1) (R - r)^2, \end{aligned} \quad (2.7)$$

provided that $r \leq a_i \leq R$ for $i = 1, \dots, n$. This implies that

$$\begin{aligned} |D_n(\mathbf{a}, \mathbf{e})| &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} [D_n(\mathbf{a}, \mathbf{a})]^{1/2} \\ &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} (R - r) \\ &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} (R - r). \end{aligned} \quad (2.8)$$

Theorem 2.2. *If $\mathbf{A} = (A_1, \dots, A_n) \in (B_p(H))^n$ and $\mathbf{B} = (B_1, \dots, B_n) \in (B_q(H))^n$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} \|D_n(\mathbf{A}, \mathbf{B})\|_1 &\leq \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p D_n(\mathbf{e}, \mathbf{V}_{\mathbf{B}, q}) \\ &\leq \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} [D_n(\mathbf{V}_{\mathbf{B}, q}, \mathbf{V}_{\mathbf{B}, q})]^{1/2} \\ &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q \\ &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q, \end{aligned} \quad (2.9)$$

where $\mathbf{V}_{\mathbf{B}, q} := \left(0, \|\Delta B_1\|_q, \dots, \sum_{k=1}^{n-1} \|\Delta B_k\|_q \right)$.

In particular, if $\mathbf{A}, \mathbf{B} \in (B_2(H))^n$, then

$$\begin{aligned} \|D_n(\mathbf{A}, \mathbf{B})\|_1 &\leq \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_2 D_n(\mathbf{e}, \mathbf{V}_{\mathbf{B}, 2}) \\ &\leq \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_2 \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \\ &\quad \times [D_n(\mathbf{V}_{\mathbf{B}, 2}, \mathbf{V}_{\mathbf{B}, 2})]^{1/2} \\ &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} \\ &\quad \times \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_2 \sum_{k=1}^{n-1} \|\Delta B_k\|_2 \\ &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_2 \sum_{k=1}^{n-1} \|\Delta B_k\|_2. \end{aligned} \quad (2.10)$$

Proof. We have by (2.4) that,

$$\begin{aligned}
 \|D_n(\mathbf{A}, \mathbf{B})\|_1 &\leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{k=i}^{j-1} \|\Delta B_k\|_q \\
 &\leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (j-i) \max_{k \in \{i, \dots, j-1\}} \|\Delta A_k\|_p \sum_{k=i}^{j-1} \|\Delta B_k\|_q \\
 &\leq \frac{1}{n^2} \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p \sum_{1 \leq i < j \leq n} (j-i) \sum_{k=i}^{j-1} \|\Delta B_k\|_q \\
 &= \frac{1}{n^2} \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p \frac{1}{2} \sum_{i,j=1}^n (j-i) \sum_{k=i}^{j-1} \|\Delta B_k\|_q \\
 &= \frac{1}{n^2} \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p \frac{1}{2} \sum_{i,j=1}^n (j-i) \left(\sum_{k=1}^{j-1} \|\Delta B_k\|_q - \sum_{k=1}^{i-1} \|\Delta B_k\|_q \right) \\
 &= \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p D_n(\mathbf{e}, \mathbf{V}_{\mathbf{B},q}).
 \end{aligned} \tag{2.11}$$

The proof of the last equality in (2.11) follows by multiplying the brackets and then summing over i and j from 1 to n .

By (2.8) we derive

$$\begin{aligned}
 (0 \leq) D_n(\mathbf{V}_{\mathbf{B},q}, \mathbf{e}) & \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} [D_n(\mathbf{V}_{\mathbf{B},q}, \mathbf{V}_{\mathbf{B},q})]^{1/2} \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} \sum_{k=1}^{n-1} \|\Delta B_k\|_q \\
 &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \sum_{k=1}^{n-1} \|\Delta B_k\|_q.
 \end{aligned} \tag{2.12}$$

By making use of (2.11) and (2.12) we derive (2.9). $\square$

More general bounds of Hölder's type for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ can be obtained as follows:

Theorem 2.3. *If $\mathbf{A} = (A_1, \dots, A_n) \in (B_p(H))^n$ and $\mathbf{B} = (B_1, \dots, B_n) \in (B_q(H))^n$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we have*

$$\begin{aligned} \|D_n(\mathbf{A}, \mathbf{B})\|_1 &\leq [D_n(\mathbf{V}_{\mathbf{A}, p, \alpha}, \mathbf{e})]^{1/\alpha} [D_n(\mathbf{V}_{\mathbf{B}, q, \beta}, \mathbf{e})]^{1/\beta} \\ &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \\ &\quad \times [D_n(\mathbf{V}_{\mathbf{A}, p, \alpha}, \mathbf{V}_{\mathbf{A}, p, \alpha})]^{1/(2\alpha)} [D_n(\mathbf{V}_{\mathbf{B}, q, \beta}, \mathbf{V}_{\mathbf{B}, q, \beta})]^{1/(2\beta)} \\ &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} \\ &\quad \times \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta \right)^{1/\beta} \\ &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta \right)^{1/\beta}, \end{aligned} \quad (2.13)$$

where

$$\mathbf{V}_{\mathbf{A}, p, \alpha} := \left(0, \|\Delta A_1\|_p^\alpha, \dots, \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \right)$$

and

$$\mathbf{V}_{\mathbf{B}, q, \beta} := \left(0, \|\Delta B_1\|_q^\beta, \dots, \sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta \right).$$

Proof. We have by (2.4) that

$$\|D_n(\mathbf{A}, \mathbf{B})\|_1 \leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{k=i}^{j-1} \|\Delta B_k\|_q. \quad (2.14)$$

By Hölder's inequality for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we have that

$$\sum_{k=i}^{j-1} \|\Delta A_k\|_p \leq (j-i)^{1/\beta} \left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha}$$

and

$$\sum_{k=i}^{j-1} \|\Delta B_k\|_q \leq (j-i)^{1/\alpha} \left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta \right)^{1/\beta},$$

which gives, by multiplication, that

$$\sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{k=i}^{j-1} \|\Delta B_k\|_q \leq (j-i) \left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta \right)^{1/\beta}.$$

Then by (2.14) we get

$$\|D_n(\mathbf{A}, \mathbf{B})\|_1 \leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (j-i) \left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta \right)^{1/\beta}. \quad (2.15)$$

By using Hölder's weighted inequality for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we get

$$\begin{aligned}
 & \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (j-i) \left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta \right)^{1/\beta} \\
 & \leq \frac{1}{n^2} \left\{ \sum_{1 \leq i < j \leq n} (j-i) \left[\left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \right]^\alpha \right\}^{1/\alpha} \left\{ \sum_{1 \leq i < j \leq n} (j-i) \left[\left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta \right)^{1/\beta} \right]^\beta \right\}^{1/\beta} \\
 & = \frac{1}{n^2} \left\{ \sum_{1 \leq i < j \leq n} (j-i) \left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha \right) \right\}^{1/\alpha} \left\{ \sum_{1 \leq i < j \leq n} (j-i) \left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta \right) \right\}^{1/\beta} \\
 & = \left\{ \frac{1}{2} \frac{1}{n^2} \sum_{i,j=1}^n (j-i) \sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha \right\}^{1/\alpha} \left\{ \frac{1}{2} \frac{1}{n^2} \sum_{i,j=1}^n (j-i) \sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta \right\}^{1/\beta} \\
 & = [D_n(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{e})]^{1/\alpha} [D_n(\mathbf{V}_{\mathbf{B},q,\beta}, \mathbf{e})]^{1/\beta}.
 \end{aligned} \tag{2.16}$$

By making use of (2.14)-(2.16) we derive the first inequality in (2.13).

By (2.8) we get

$$\begin{aligned}
 D_n(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{e}) & \leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} [D_n(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{V}_{\mathbf{A},p,\alpha})]^{1/2} \\
 & \leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \\
 & \leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha,
 \end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
 D_n(\mathbf{V}_{\mathbf{B},q,\beta}, \mathbf{e}) & \leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} [D_n(\mathbf{V}_{\mathbf{B},q,\beta}, \mathbf{V}_{\mathbf{B},q,\beta})]^{1/2} \\
 & \leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} \sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta \\
 & \leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta.
 \end{aligned} \tag{2.18}$$

Therefore

$$\begin{aligned}
& [D_n(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{e})]^{1/\alpha} [D_n(\mathbf{V}_{\mathbf{B},q,\beta}, \mathbf{e})]^{1/\beta} \\
& \leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} [D_n(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{V}_{\mathbf{A},p,\alpha})]^{1/(2\alpha)} [D_n(\mathbf{V}_{\mathbf{B},q,\beta}, \mathbf{V}_{\mathbf{B},q,\beta})]^{1/(2\beta)} \\
& \leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) \right]^{1/2} \\
& \times \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta \right)^{1/\beta} \\
& \leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta \right)^{1/\beta},
\end{aligned}$$

which proves the second part of (2.13). $\square$

Remark 2.1. *If*

$$\mathbf{V}_{\mathbf{A},2,\alpha} := \left(0, \|\Delta A_1\|_2^\alpha, \dots, \sum_{k=1}^{n-1} \|\Delta A_k\|_2^\alpha \right) \text{ and } \mathbf{V}_{\mathbf{B},2,\beta} := \left(0, \|\Delta B_1\|_2^\beta, \dots, \sum_{k=1}^{n-1} \|\Delta B_k\|_2^\beta \right),$$

then for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we have

$$\begin{aligned}
\|D_n(\mathbf{A}, \mathbf{B})\|_1 & \leq [D_n(\mathbf{V}_{\mathbf{A},2,\alpha}, \mathbf{e})]^{1/\alpha} [D_n(\mathbf{V}_{\mathbf{B},2,\beta}, \mathbf{e})]^{1/\beta} \\
& \leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \\
& \times [D_n(\mathbf{V}_{\mathbf{A},2,\alpha}, \mathbf{V}_{\mathbf{A},2,\alpha})]^{1/(2\alpha)} [D_n(\mathbf{V}_{\mathbf{B},2,\beta}, \mathbf{V}_{\mathbf{B},2,\beta})]^{1/(2\beta)} \\
& \leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) \right]^{1/2} \\
& \times \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_2^\alpha \right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_2^\beta \right)^{1/\beta} \\
& \leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_2^\alpha \right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_2^\beta \right)^{1/\beta}.
\end{aligned} \tag{2.19}$$

Now, if we take $\alpha = p$ and $\beta = q$ in (2.13), then we get

$$\begin{aligned}
 \|D_n(\mathbf{A}, \mathbf{B})\|_1 &\leq [D_n(\mathbf{V}_{\mathbf{A}, p, p}, \mathbf{e})]^{1/p} [D_n(\mathbf{V}_{\mathbf{B}, q, q}, \mathbf{e})]^{1/q} \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \\
 &\quad \times [D_n(\mathbf{V}_{\mathbf{A}, p, p}, \mathbf{V}_{\mathbf{A}, p, p})]^{1/(2p)} [D_n(\mathbf{V}_{\mathbf{B}, q, q}, \mathbf{V}_{\mathbf{B}, q, q})]^{1/(2q)} \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} \\
 &\quad \times \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^p \right)^{1/p} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q^q \right)^{1/q} \\
 &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^p \right)^{1/p} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q^q \right)^{1/q}.
 \end{aligned} \tag{2.20}$$

By using the trace properties, we can state that

$$\begin{aligned}
 |\operatorname{tr}(D_n(\mathbf{A}, \mathbf{B}))| & \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} \\
 &\quad \times \left[\operatorname{tr} \left(\sum_{k=1}^{n-1} |\Delta A_k|^p \right) \right]^{1/p} \left[\operatorname{tr} \left(\sum_{k=1}^{n-1} |\Delta B_k|^q \right) \right]^{1/q} \\
 &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left[\operatorname{tr} \left(\sum_{k=1}^{n-1} |\Delta A_k|^p \right) \right]^{1/p} \left[\operatorname{tr} \left(\sum_{k=1}^{n-1} |\Delta B_k|^q \right) \right]^{1/q},
 \end{aligned} \tag{2.21}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, for $p = q = 2$ we get the trace inequality:

$$\begin{aligned}
 |\operatorname{tr}(D_n(\mathbf{A}, \mathbf{B}))| & \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \right]^{1/2} \\
 &\quad \times \left[\operatorname{tr} \left(\sum_{k=1}^{n-1} |\Delta A_k|^2 \right) \right]^{1/2} \left[\operatorname{tr} \left(\sum_{k=1}^{n-1} |\Delta B_k|^2 \right) \right]^{1/2} \\
 &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left[\operatorname{tr} \left(\sum_{k=1}^{n-1} |\Delta A_k|^2 \right) \right]^{1/2} \left[\operatorname{tr} \left(\sum_{k=1}^{n-1} |\Delta B_k|^2 \right) \right]^{1/2}.
 \end{aligned} \tag{2.22}$$

3. p -SCHATTEN NORM INEQUALITIES

We also have:

Theorem 3.1. If $\mathbf{A} = (A_1, \dots, A_n)$ and $\mathbf{B} = (B_1, \dots, B_n) \in (B_p(H))^n$ for $p \geq 1$, then

$$\begin{aligned} \|D_n(\mathbf{A}, \mathbf{B})\|_p &\leq D_n(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},p}) \\ &\leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_p \\ &\leq \frac{1}{4} \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_p, \end{aligned} \quad (3.1)$$

where $\mathbf{V}_{\mathbf{A},p} := \left(0, \|\Delta A_1\|_p, \dots, \sum_{k=1}^{n-1} \|\Delta A_k\|_p \right)$ and $\mathbf{V}_{\mathbf{B},p} := \left(0, \|\Delta B_1\|_p, \dots, \sum_{k=1}^{n-1} \|\Delta B_k\|_p \right)$.

Proof. As in the proof of Theorem 2.1 we have, by utilising property (1.13), that

$$\begin{aligned} &\|D_n(\mathbf{A}, \mathbf{B})\|_p \\ &= \frac{1}{n^2} \left\| \sum_{1 \leq i < j \leq n} \sum_{k=i}^{j-1} \Delta A_k \sum_{k=i}^{j-1} \Delta B_k \right\|_p \leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \left\| \sum_{k=i}^{j-1} \Delta A_k \sum_{k=i}^{j-1} \Delta B_k \right\|_p \\ &\leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \left\| \sum_{k=i}^{j-1} \Delta A_k \right\|_p \left\| \sum_{k=i}^{j-1} \Delta B_k \right\|_p \\ &\leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{k=i}^{j-1} \|\Delta B_k\|_p \\ &= \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \left(\sum_{k=1}^{j-1} \|\Delta A_k\|_p - \sum_{k=1}^{i-1} \|\Delta A_k\|_p \right) \left(\sum_{k=1}^{j-1} \|\Delta B_k\|_p - \sum_{k=1}^{i-1} \|\Delta B_k\|_p \right) \\ &= \frac{1}{2n^2} \sum_{i,j=1}^n \left(\sum_{k=1}^{j-1} \|\Delta A_k\|_p - \sum_{k=1}^{i-1} \|\Delta A_k\|_p \right) \left(\sum_{k=1}^{j-1} \|\Delta B_k\|_p - \sum_{k=1}^{i-1} \|\Delta B_k\|_p \right) \\ &= D_n(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},p}). \end{aligned} \quad (3.2)$$

The rest follows as in the proof of Theorem 2.1 and we omit the details. $\square$

By utilising a similar argument to the one in the proof of Theorem 2.2 we can state

Theorem 3.2. *If $\mathbf{A} = (A_1, \dots, A_n)$ and $\mathbf{B} = (B_1, \dots, B_n) \in (B_p(H))^n$ for $p \geq 1$, then*

$$\begin{aligned}
 \|D_n(\mathbf{A}, \mathbf{B})\|_p &\leq \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p D_n(\mathbf{e}, \mathbf{V}_{\mathbf{B}, p}) \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p [D_n(\mathbf{V}_{\mathbf{B}, p}, \mathbf{V}_{\mathbf{B}, p})]^{1/2} \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) \right]^{1/2} \\
 &\quad \times \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_p \\
 &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \max_{k \in \{1, \dots, n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_p.
 \end{aligned} \tag{3.3}$$

Finally, we have:

Theorem 3.3. *If $\mathbf{A} = (A_1, \dots, A_n)$ and $\mathbf{B} = (B_1, \dots, B_n) \in (B_p(H))^n$ for $p \geq 1$, then for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we have*

$$\begin{aligned}
 \|D_n(\mathbf{A}, \mathbf{B})\|_p &\leq [D_n(\mathbf{V}_{\mathbf{A}, p, \alpha}, \mathbf{e})]^{1/\alpha} [D_n(\mathbf{V}_{\mathbf{B}, p, \beta}, \mathbf{e})]^{1/\beta} \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \\
 &\quad \times [D_n(\mathbf{V}_{\mathbf{A}, p, \alpha}, \mathbf{V}_{\mathbf{A}, p, \alpha})]^{1/(2\alpha)} [D_n(\mathbf{V}_{\mathbf{B}, p, \beta}, \mathbf{V}_{\mathbf{B}, p, \beta})]^{1/(2\beta)} \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) \right]^{1/2} \\
 &\quad \times \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_p^\beta \right)^{1/\beta} \\
 &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_p^\beta \right)^{1/\beta},
 \end{aligned} \tag{3.4}$$

where

$$\mathbf{V}_{\mathbf{A}, p, \alpha} := \left(0, \|\Delta A_1\|_p^\alpha, \dots, \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \right), \quad \mathbf{V}_{\mathbf{B}, p, \beta} := \left(0, \|\Delta B_1\|_p^\beta, \dots, \sum_{k=1}^{n-1} \|\Delta B_k\|_p^\beta \right).$$

We observe that for $\alpha = \beta = 2$ we derive the inequality

$$\begin{aligned}
 \|D_n(\mathbf{A}, \mathbf{B})\|_p &\leq [D_n(\mathbf{V}_{\mathbf{A},p,2}, \mathbf{e})]^{1/2} [D_n(\mathbf{V}_{\mathbf{B},p,2}, \mathbf{e})]^{1/2} \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \\
 &\quad \times [D_n(\mathbf{V}_{\mathbf{A},p,2}, \mathbf{V}_{\mathbf{A},p,2})]^{1/4} [D_n(\mathbf{V}_{\mathbf{B},p,2}, \mathbf{V}_{\mathbf{B},p,2})]^{1/4} \\
 &\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) \right]^{1/2} \\
 &\quad \times \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^2 \right)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_p^2 \right)^{1/2} \\
 &\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^2 \right)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_p^2 \right)^{1/2}.
 \end{aligned} \tag{3.5}$$

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