CANADIAN TRANSACTIONS of FUNCTIONAL ANALYSIS
Volume 1, Issue 1, (2025) Article 250105

MATRIX INEQUALITIES FOR SECTOR MATRICES: A COMPREHENSIVE REVIEW OF DEVELOPMENTS AND GENERALIZATIONS

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Abstract. In recent years, the notion of sector matrices, which are natural extensions of positive definite Hermitian matrices, has played a pivotal role in the development of various matrix inequalities. In this review article, we present a unified perspective on key advances in this area, from the early results of Lin and Drury on matrix inequalities to the recent work of Khosravi, Malekinejad, and their collaborators concerning operator means and the role of the sectorial angle in generalized constants. In particular, we focus on how classical inequalities, such as Pólya type inequalities and other operator mean related inequalities, are extended and proved within the framework of sector matrices. We also examine in detail the monotonicity and order reversing properties of operator means under the real part mapping, as well as the sharp appearance of the factor $\sec^2 \alpha$ in such generalizations. By consolidating existing results, this article aims to provide a coherent reference and outline promising directions for future research in this active field.

2020 Mathematics Subject Classification: 15A60, 15B48, 47A64, 47A63.
Keywords: Sector matrices, Hadamard product, Positive linear map, Operator monotone function, Operator mean.

1. INTRODUCTION AND PRELIMINARY

Let $\mathbb{M}_{m\times n}$ be the space of all $m \times n$ matrices with entries in the complex field $\mathbb{C}$. Let $\mathbb{M}_n$ denote the set of $n \times n$ complex matrices. For a Hermitian matrix $A \in \mathbb{M}_n$, we write $A \geq 0$ if A is positive semidefinite, i.e., $\langle Ax, x\rangle \geq 0$ for all vectors $x \in \mathbb{C}^n$. We also write $A > 0$ if A is positive definite, i.e., it is positive semidefinite and invertible.

A matrix $A \in \mathbb{M}_n$ is called accretive if $\mathcal{R}A$ is positive in its Cartesian (or Toeplitz) decomposition, $A = \mathcal{R}A + i\mathcal{I}A$, where $\mathcal{R}A = \frac{A+A^*}{2}$ and $\mathcal{I}A = \frac{A-A^*}{2i}$. The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

Recieved: August 2025, Accepted: December 2025.

It is well known that $W(A)$ is a compact and convex set in $\mathbb{C}$ and contains the spectrum of A. For more information see [19].

We say $A \in \mathbb{M}_n$ is accretive if $\mathcal{R}A > 0$. When researching accretive matrices, it is necessary to consider sectorial types. For this purpose, let $\mathcal{S}_\alpha$ denote the sector region in the complex plane $\mathbb{C}$ as follows:

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : \mathcal{R}z > 0, |\mathcal{I}z| \leq (\mathcal{R}z)\tan\alpha\},$$

where $0 \leq \alpha < \frac{\pi}{2}$. A matrix $A \in \mathbb{M}_n$ is called a sectorial matrix if $W(A) \subset \mathcal{S}_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$. We simply write $A \in \mathcal{S}_\alpha$ if $W(A) \subset \mathcal{S}_\alpha$. Clearly, A is positive definite if and only if $W(A) \subset \mathcal{S}_0$. We refer the reader to [1,3,4,6,27,33,34] as a sample of articles treating this topic.

The following lemma states the relations between the real part of the inverse of A and the inverse of $\mathcal{R}A$ for sectorial operators.

Lemma 1.1. *([24], [25]) If $A \in \mathcal{S}_\alpha$, then so is A^{-1} and*

$$\mathcal{R}(A^{-1}) \leq \mathcal{R}^{-1}(A) \leq \sec^2\alpha\ \mathcal{R}(A^{-1}).$$

An operator mean σ in the sense of Kubo–Ando [22] is defined by an operator monotone function $f : (0,\infty) \to (0,\infty)$ with $f(1) = 1$ as

$$A\sigma B = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2},$$

for positive invertible operators A and B. Here the function f is called the representing function of σ. Recently, Bedrani et al. in [3] proved that this definition can also be used for accretive operators. Some important operator means, introduced in the seminal work of Kubo and Ando [22] and extensively studied in [7], are given as follows:

- Arithmetic mean: $A\nabla B = (A+B)/2$ and t-weighted arithmetic mean: $A\nabla_t B = tA + (1-t)B$. $(0 < t < 1)$
- Geometric mean: $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ and t-weighted geometric mean: $A\sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$. $(0 < t < 1)$
- Heinz mean: $H_t(A,B) = (A\sharp_t B + B\sharp_t A)/2$. $(0 < t < 1)$
- Heron mean: $F_t(A,B) = t(A\nabla B) + (1-t)A\sharp B$. $(0 < t < 1)$
- Logarithmic mean: $L(A,B) = \int_0^1 A\sharp_t B\, dt$.

Let σ be an operator mean with representing function f. The operator mean with representing function $f(t^{-1})^{-1}$ is called the adjoint of σ and denoted by σ^*. The definition results in:

$$A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1}.$$

For two operator means σ_f and σ_g with representing functions f and g, it is written $\sigma \leq \tau$ if $A\sigma B \leq A\tau B$ for every two positive operators A and B, or equivalently if $f(t) \leq g(t)$ for all $t > 0$.

We present the following definition for a class of operator means.

Definition 1.2. *[27] An operator mean σ with a representing function f is called a semi-selfadjoint mean if $\sigma^* \leq \sigma$, which is equivalent to each of the following conditions:*

- $f(t)f(\frac{1}{t}) \geq 1$,
- $(A\sigma B)^{-1} \leq A^{-1}\sigma B^{-1}$.

One can easily see that if σ is semi-selfadjoint and $\sigma \leq \tau$, then τ is semi-selfadjoint too. Since the geometric mean and the t-weighted geometric mean are semi-selfadjoint, it follows that the t-weighted arithmetic mean, Heinz mean, Heron mean and logarithmic mean are also semi-selfadjoint.

Let J be an interval in $\mathbb{R}$. A continuous function $f : J \longrightarrow \mathbb{R}$ is super multiplicative on J if $f(xy) \geq f(x)f(y)$ for every $x, y \in J$. The function f is said to be matrix monotone if $A \geq B$ with spectra contained in J implies $f(A) \geq f(B)$. Also, f is called matrix concave if

$$f(\lambda A + (1-\lambda)B) \geq \lambda f(A) + (1-\lambda)f(B)$$

for all $\lambda \in [0,1]$ and for every Hermitian matrix $A, B \in \mathbb{M}_n$ whose spectra are in the interval J.

Notice that if f is a nonnegative continuous function on $[0,\infty)$, then f is matrix monotone if and only if f is matrix concave; see [33, Corollary 1.12]. Also, we define the following notation:

$$\mathbf{m} = \{f \mid f : (0,\infty) \to (0,\infty) \text{ is a matrix monotone function with } f(1) = 1\}.$$

2. Operator Mean Inequalities for Sector Matrices and Their Extension

In 2014, Drury [12] extended the definition of the geometric mean of two positive matrices to sector matrices, defined for $A, B \in M_n$ as

$$A\sharp B = \left(\frac{2}{\pi}\int_0^\infty (tA + t^{-1}B)^{-1}\frac{dt}{t}\right)^{-1}. \tag{2.1}$$

It was shown in that paper that when A and B are positive, this definition coincides with the classical one for positive definite matrices.

Raïssouli et al. [34] introduced the weighted geometric mean for two sector matrices $A, B \in M_n$ as

$$\begin{aligned} A\sharp_t B &= \frac{\sin(t\pi)}{\pi}\int_0^\infty s^{t-1}(A^{-1} + sB^{-1})^{-1}ds \qquad (2.2)\\ &= \frac{\sin(t\pi)}{\pi}\int_0^1 \frac{s^{t-1}}{(1-s)^t}(A!_s B)ds. \end{aligned}$$

In the same paper, the authors showed that for $t = \frac{1}{2}$, Drury's definition (2.1) coincides with definition (2.2). Moreover, if $A, B > 0$, then from (2.2) we obtain for every $0 < t < 1$,

$$A\sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \tag{2.3}$$

which is the standard definition of the weighted geometric mean for positive definite matrices. In 2020, Kittaneh et al. [3] showed that this definition also holds for sector matrices.

In 2016, Lin et al. [23] first studied the relation between $\mathcal{R}A\sigma\mathcal{R}B$ and $\mathcal{R}(A\sigma B)$ for $\sigma = \sharp$ and $\sigma =!$, and obtained the following:

If $A, B \in S_\alpha$, then

$$\mathcal{R}(A!B) \geq \mathcal{R}(A)!\mathcal{R}(B), \tag{2.4}$$

and

$$\mathcal{R}(A\sharp B) \geq \mathcal{R}(A)\sharp\mathcal{R}(B). \tag{2.5}$$

Next, for $A, B \in S_\alpha$ and $0 < t < 1$, inequalities (2.4) and (2.5) were extended and their reverse forms were obtained.

In 2017, Raïssouli et al. [34] established the following inequalities, which are extensions of inequalities (2.4) and (2.5).

$$\mathcal{R}(A!_t B) \geq \mathcal{R}A!_t\mathcal{R}B. \tag{2.6}$$

and

$$\mathcal{R}(A\sharp_t B) \geq \mathcal{R}A\sharp_t\mathcal{R}B. \tag{2.7}$$

Lin [25] obtained in 2016 the reverse of inequality (2.6):

$$\mathcal{R}(A!_t B) \leq \sec^2\alpha\, \left(\mathcal{R}(A)!_t\mathcal{R}(B)\right). \tag{2.8}$$

In 2019, Tan et al. [38] proved the following reverse inequality of (2.7).

$$\mathcal{R}(A\sharp_t B) \leq \sec^2\alpha\,(\mathcal{R}A\sharp_t\mathcal{R}B). \tag{2.9}$$

Therefore, in general we have:

$$\mathcal{R}A\sharp_t\mathcal{R}B \leq \mathcal{R}(A\sharp_t B) \leq \sec^2\alpha\,(\mathcal{R}A\sharp_t\mathcal{R}B). \tag{2.10}$$

An inequality analogous to (2.10) for the logarithmic mean was proved by Tan et al. [38] for $A, B \in S_\alpha$.

$$L(\mathcal{R}(A), \mathcal{R}(B)) \leq \mathcal{R}L(A, B) \leq \sec^2\alpha\, L(\mathcal{R}(A), \mathcal{R}(B)). \tag{2.11}$$

For the Heinz mean, Mao et al. [29] established inequality (2.11) for $A, B \in S_\alpha$, as given below.

$$H_t(\mathcal{R}(A), \mathcal{R}(B)) \leq \mathcal{R}H_t(A, B) \leq \sec^2\alpha\, H_t(\mathcal{R}(A), \mathcal{R}(B)). \tag{2.12}$$

The following inequality for the Heron mean, valid for $A, B \in S_\alpha$, was obtained by Ghazanfari et al. [17]:

$$F_t(\mathcal{R}A, \mathcal{R}B) \leq \mathcal{R}F_t(A, B) \leq \sec^2\alpha\, F_t(\mathcal{R}A, \mathcal{R}B). \tag{2.13}$$

Finally, Bedrani et al. [3] in 2021 generalized inequalities (2.4)–(2.13) for $A, B \in S_\alpha$, as stated in the following lemma.

Lemma 2.1. *[3] Let $A, B \in \mathcal{S}_\alpha$. Then $A\sigma B \in \mathcal{S}_\alpha$ and*

$$\mathcal{R}A\sigma\mathcal{R}B \leq \mathcal{R}(A\sigma B) \leq \sec^2\alpha(\mathcal{R}A\sigma\mathcal{R}B). \tag{2.14}$$

Kubo and Ando [22] obtained the following inequality for positive definite matrices: If $A, B \in M_n$, $f \in \mathbf{m}$, then

$$A!B \leq A\sigma_f B \leq A\nabla B. \tag{2.15}$$

This inequality has been extended to sector matrices by many mathematicians in special cases. Raïssouli et al. remarked in [34] that this inequality cannot hold for sector matrices in general.

In inequality (2.15), if $t = \frac{1}{2}$ and $\sigma_f = \sharp$, the right-hand side is the well-known AM-GM inequality and the left-hand side is the GM-HM inequality.

Lin in [25] obtained the following extension of the AM-GM inequality for sector matrices:

$$\mathcal{R}(A\sharp B) \leq \sec^2\alpha\,\mathcal{R}(A\nabla B); \tag{2.16}$$

and the following inequality was proved by Liu et al. [26], which is an extension of the GM-HM inequality. If $A, B \in S_\alpha$, then

$$\mathcal{R}(A!B) \leq \sec^2 \alpha\, \mathcal{R}(A\sharp B). \tag{2.17}$$

Bedrani et al. [3, Theorem 4.1] extended inequalities (2.16) and (2.17) for arbitrary matrix means as follows: If $A, B \in S_\alpha$, $f \in \mathbf{m}$, and $f'(1) = 1$, then for some $t \in (0, 1)$

$$\cos^2 \alpha\, \mathcal{R}(A!_t B) \leq \mathcal{R}(A\sigma_f B) \leq \sec^2 \alpha\, \mathcal{R}(A\nabla_t B). \tag{2.18}$$

Mao [29, Theorem 2.9] proved the following double inequality involving the geometric mean $\sharp$ and the Heinz mean H_t:

$$\cos^2 \alpha\, \mathcal{R}(A\sharp B) \leq \mathcal{R}H_t(A, B) \leq \sec^2 \alpha\, \mathcal{R}\left(\frac{A+B}{2}\right), \tag{2.19}$$

for all $A, B \in \mathcal{S}_\alpha$ and $0 \leq t \leq 1$. Similarly, Tan and Xie [38, Theorem 3.5] established an analogous inequality for the logarithmic mean $\mathcal{L}_t$:

$$\cos^2 \alpha\, \mathcal{R}(A\sharp B) \leq \mathcal{R}\mathcal{L}_t(A, B) \leq \sec^2 \alpha\, \mathcal{R}\left(\frac{A+B}{2}\right), \tag{2.20}$$

under the same conditions on A, B, and t.
Finally, a significant advancement was made by Malekinejad et al. in [27] in 2020, who established refined extensions of matrix inequalities within the framework of sectorial matrices. Specifically, they proved that for any two semi-selfadjoint operator means σ_1 and σ_2 with $\sigma_1 \leq \sigma_2$, and for all $A, B \in \mathcal{S}_\alpha$, the following inequalities hold:

$$\mathcal{R}(A\sigma_1 B) \leq \sec^2 \alpha\, \mathcal{R}(A\sigma_2 B), \tag{2.21}$$

$$\mathcal{R}(A\sigma_2 B)^{-1} \leq \sec^2 \alpha\, \mathcal{R}(A\sigma_1 B)^{-1}. \tag{2.22}$$

These results generalize several earlier inequalities in the same setting and provide a unified perspective on monotonicity and order-reversing behavior of operator means under the real part map on sectorial matrices. Inequalities (2.18), (2.19), and (2.20) are unified and generalized by those in (2.21), which hold for arbitrary semi-selfadjoint operator means satisfying $\sigma_1 \leq \sigma_2$. Consequently, (2.21) extends [29, Theorem 2.9], [38, Theorem 3.5], and [3, Theorem 4.1].

In view of the inequalities in (2.21), sectorial analogues of many classical inequalities between operator means can be derived. For instance, the results in [3, Propositions 3.2-3.3, Theorems 3.3, 5.2], [26, Theorem 1.2], [29, Theorem 2.9], and [38, Theorem 3.5] all arise as special cases of these inequalities. Moreover, these inequalities improve and extend [39, Theorem 2.4].
A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ is called positive if $\Phi(A) \geq 0$ when $A \geq 0$. If $\Phi(I) = I$, where I denotes the identity matrix, then Φ is called unital. Notice that for any positive unital linear map Φ and any matrix $A \in \mathbb{M}_n$, the real part commutes with Φ; that is,

$$\Phi(\mathcal{R}A) = \mathcal{R}(\Phi(A)). \tag{2.23}$$

The authors proved the following inequalities: for $A, B \in \mathcal{S}_\alpha$, any unital positive linear map Φ, and any two semi-selfadjoint operator means $\sigma_1 \leq \sigma_2$,

$$\Phi\big(\mathcal{R}(A\sigma_1 B)^{-1}\big) \leq \sec^2 \alpha\, \mathcal{R}\big(\Phi(A^{-1})\sigma_2 \Phi(B^{-1})\big)$$

and

$$\mathcal{R}\big((\Phi(A)\sigma_1\Phi(B))^{-1}\big) \leq \sec^2\alpha\,\mathcal{R}\big(\Phi(A^{-1})\sigma_2\Phi(B^{-1})\big),$$

which are established in [27, Theorems 2.4 and 2.6], respectively. These results generalize the inequalities of Mao [29, Theorem 2.13], Lin [25, Theorem 3], and Yan [42, Theorem 6, Theorem 7].

The operator norm $\|A\|$ of $A \in \mathbb{M}_n$ is defined by

$$\|A\| = \sup\{|\langle Ax, y\rangle| : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1\}.$$

Recall that a norm $||\cdot||$ on $\mathbb{M}_n$ is unitarily invariant if $||UAV|| = ||A||$ for any $A \in \mathbb{M}_n$ and for all unitary matrices $U, V \in \mathbb{M}_n$.

In the sequel, we present two useful lemmas that are needed in the proofs of our main results.

Lemma 2.2. *[7, 12] Let $A \in \mathbb{M}_n$. Then*

$$\lambda_j(\mathcal{R}A) \leq \sigma_j(A) \leq \sec^2\alpha\,\lambda_j(\mathcal{R}A), \qquad j = 1, \ldots, n. \tag{2.24}$$

Lemma 2.3. *[7, 44] Let $A \in \mathcal{S}_\alpha$. Then*

$$||\mathcal{R}A|| \leq ||A|| \leq \sec(\alpha)\,||\mathcal{R}A||$$

for any unitarily invariant norm $||\cdot||$ on $\mathbb{M}_n$.

Moreover, the following inequality is established in [27, Theorem 2.8]: for all $A, B \in \mathcal{S}_\alpha$, any unital positive linear map Φ, and any two semi-selfadjoint operator means $\sigma_1 \leq \sigma_2$,

$$\mathcal{R}^{-1}\,\Phi(A\sigma_1 B) \leq \sec^2\alpha\,\mathcal{R}\,\Phi(A^{-1}\sigma_2 B^{-1}),$$

which generalizes [42, Theorem 10].

Furthermore, the authors obtained norm versions of the above inequalities. Specifically, in [27, Theorem 2.14], they proved that for all $A, B \in \mathcal{S}_\alpha$, any unital positive linear map Φ, and any two semi-selfadjoint operator means $\sigma_1 \leq \sigma_2$, the following norm inequalities hold:

$$\begin{aligned}
&\|\Phi(A\sigma_1 B)^{-1}\| \leq \sec^3\alpha\,\|\Phi(A^{-1})\sigma_2\Phi(B^{-1})\|,\\
&\|(\Phi(A)\sigma_1\Phi(B))^{-1}\| \leq \sec^3\alpha\,\|\Phi(A^{-1})\sigma_2\Phi(B^{-1})\|,\\
&\|\Phi^{-1}(A\sigma_1 B)\| \leq \sec^3\alpha\,\|\Phi(A^{-1})\sigma_2\Phi(B^{-1})\|,\\
&\|\Phi^{-1}(A\sigma_1 B)\| \leq \sec^3\alpha\,\|\Phi(A^{-1}\sigma_2 B^{-1})\|,
\end{aligned}$$

where $\|\cdot\|$ denotes any unitarily invariant norm. These results provide the norm counterparts of [27, Theorems 2.4, 2.6, 2.8, and 2.9].

Notably, Mao [29, Theorem 2.12] proved the following double inequality for the geometric mean $\sharp$ and the Heinz mean H_t:

$$\cos^3\alpha\,\|A\sharp B\| \leq \|H_t(A, B)\| \leq \sec^3\alpha\,\left\|\frac{A+B}{2}\right\|,$$

for all $A, B \in \mathcal{S}_\alpha$ and $0 \leq t \leq 1$. [27, Theorem 2.14] generalizes this result, as the above inequality arises as a special case by choosing appropriate means and setting $\Phi(X) = X$.

In addition, the authors established singular value versions of the aforementioned inequalities. Specifically, in [27, Theorem 2.18] they showed that for all $A, B \in \mathcal{S}_\alpha$, any unital positive linear map Φ, and any two semi-selfadjoint operator means $\sigma_1 \leq \sigma_2$, the following inequalities hold for all j:

$$\begin{aligned}
&s_j\big(\Phi^{-1}(A\sigma_1 B)\big) \leq \sec^4\alpha\, s_j\big(\Phi(A^{-1})\sigma_2\Phi(B^{-1})\big)\,,\\
&s_j\big(\Phi^{-1}(A\sigma_1 B)\big) \leq \sec^4\alpha\, s_j\big(\Phi(A^{-1}\sigma_2 B^{-1})\big)\,,\\
&s_j\big(\Phi(A\sigma_1 B)^{-1}\big) \leq \sec^4\alpha\, s_j\big(\Phi(A^{-1})\sigma_2\Phi(B^{-1})\big)\,,\\
&s_j\big((\Phi(A)\sigma_1\Phi(B))^{-1}\big) \leq \sec^4\alpha\, s_j\big(\Phi(A^{-1})\sigma_2\Phi(B^{-1})\big)\,,
\end{aligned}$$

where $s_j(\cdot)$ denotes the j-th singular value. These results constitute the singular value counterparts of [27, Theorems 2.4, 2.6, 2.8, and 2.9]. An inequality for the geometric mean $\sharp$ and the Heinz mean H_t was established by Mao [29, Theorem 2.10]:

$$\cos^4\alpha\, s_j(A\sharp B) \leq s_j\big(H_t(A,B)\big) \leq \sec^4\alpha\, s_j\left(\frac{A+B}{2}\right),$$

for all $A, B \in \mathcal{S}_\alpha$ and $0 \leq t \leq 1$. [27, Theorem 2.18] generalizes this result, as the above double inequality arises as a special case by appropriate choices of the operator means and taking $\Phi(X) = X$.

Finally, the authors also derived determinant versions of the established inequalities. Specifically, in [27, Theorem 2.22], they proved that for all $A, B \in \mathcal{S}_\alpha$, any unital positive linear map Φ, and any two semi-selfadjoint operator means $\sigma_1 \leq \sigma_2$, the following determinant inequalities hold:

$$\begin{aligned}
&\big|\det\big(\Phi^{-1}(A\sigma_1 B)\big)\big| \leq \sec^{3n}\alpha\,\big|\det\big(\Phi(A^{-1}\sigma_2 B^{-1})\big)\big|,\\
&\big|\det\big(\Phi^{-1}(A\sigma_1 B)\big)\big| \leq \sec^{3n}\alpha\,\big|\det\big(\Phi(A^{-1})\sigma_2\Phi(B^{-1})\big)\big|,\\
&\big|\det\big((\Phi(A)\sigma_1\Phi(B))^{-1}\big)\big| \leq \sec^{3n}\alpha\,\big|\det\big(\Phi(A^{-1})\sigma_2\Phi(B^{-1})\big)\big|,\\
&\big|\det\big(\Phi(A\sigma_1 B)^{-1}\big)\big| \leq \sec^{3n}\alpha\,\big|\det\big(\Phi(A^{-1})\sigma_2\Phi(B^{-1})\big)\big|.
\end{aligned}$$

These results provide the determinant counterparts of [27, Theorems 2.4, 2.6, 2.8, and 2.9]. Yang and Mao [42, Theorem 9] proved the following double inequality for the geometric mean $\sharp$ and the Heinz mean H_t:

$$\cos^{3n}\alpha\,\big|\det(A\sharp B)\big| \leq \big|\det H_t(A,B)\big| \leq \sec^{3n}\alpha\,\left|\det\left(\frac{A+B}{2}\right)\right|,$$

for all $A, B \in \mathcal{S}_\alpha$ and $0 \leq t \leq 1$. [27, Theorem 2.22] extends this result, as the above inequality is recovered by choosing appropriate operator means and setting $\Phi(X) = X$. The authors further established reverse-type inequalities by interchanging the roles of the operator means σ_1 and σ_2. These results are presented in operator, norm, determinant, and singular value forms, featuring refined angular constants such as $\sec^4\alpha$, $\sec^5\alpha$, and $\sec^6\alpha$. In the special case of accretive-dissipative matrices, these constants reduce to explicit numerical bounds, namely $4\sqrt{2}$ and 8. Finally, it is shown that these reverse inequalities generalize and extend earlier results, including those of Liu and Wang.

It is well known that the numerical radius $\omega(A)$ of $A \in \mathbb{M}_n$ is defined by

$$\omega(A) = \sup\{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}.$$

When $A \in S_0$, we have $\omega(A) = \|A\|$. Therefore

$$\omega(\mathcal{R}A) = \|\mathcal{R}A\|. \tag{2.25}$$

Bedrani et al. [3] showed that if $A \in S_\alpha$, then

$$\omega(\mathcal{R}A) \le \omega(A) \le \sec\alpha\ \omega(\mathcal{R}A). \tag{2.26}$$

In the setting of numerical range, the authors also obtained the following inequality: for any two semi-selfadjoint operator means $\sigma_1 \le \sigma_2$ and all $A, B \in \mathcal{S}_\alpha$,

$$\omega(A\sigma_1 B) \le \sec^3\alpha\, \omega(A\sigma_2 B),$$

which is established in [27, Theorem 3.2]. This result extends and refines the double inequality of Kittaneh et al. [4, Theorem 3.10] for the geometric and Heinz means, namely

$$\cos^4\alpha\, \omega(A\sharp B) \le \omega(H_t(A,B)) \le \sec^4\alpha\, \omega\left(\frac{A+B}{2}\right),$$

by providing sharper bounds with exponent 3 (instead of 4) and a unified framework for arbitrary semi-selfadjoint means.

3. Sectorial Extensions of Operator Mean Inequalities with Kantorovich Constants

Khosravi et al. in [20] proved that if $0 < mI \le A, B \le MI$, then for every positive unital linear map Φ and operator means σ_1 and σ_2 between some σ and σ^*,

$$\Phi^p(A\sigma B) \le \alpha^p \Phi^p(A\sigma^* B),$$

where $p > 0$, $\alpha = \max\{K, 4^{1-\frac{2}{p}}K\}$ and $K = \frac{(M+m)^2}{4mM}$ is the Kantorovich constant.

In addition, they showed that if σ_1 and σ_2 are between ∇_t and $!_t$, then

$$f\left(\Phi(A)\right)\sigma_1 f\left(\Phi(B)\right) \le K f\left(\Phi(A\sigma_2 B)\right), \tag{3.1}$$

for every nonzero operator monotone function f on $[0,\infty)$. Khosravi et al. [21] established sectorial analogues of several classical inequalities. We now introduce and examine these inequalities in this section. A fundamental result in the theory of operator means is the well-known Choi's inequality: for any positive operator $A \in B(H)$ and every unital positive linear map Φ, one has

$$\Phi^{-1}(A) \le \Phi(A^{-1}).$$

This inequality, originally due to Choi [33], plays a key role in the study of operator monotone functions and matrix inequalities.

Khosravi et al. [21] employed Choi's inequality to prove the following proposition, which played a pivotal role in establishing their main results.

Proposition 3.1. *[21] Let $A, B \in \mathcal{S}_\alpha$ for some $0 \leq \alpha \leq \pi/2$ such that $0 < mI \leq \mathcal{R}A, \mathcal{R}B \leq MI$. If σ is an arbitrary mean and σ_1, σ_2 are two operator means between σ and σ^*, then*

$$\cos^2 \alpha \Phi \mathcal{R}(A\sigma_1 B) + mM\Phi^{-1}\mathcal{R}(A\sigma_2 B) \leq (M+m)I \tag{3.2}$$

for every positive unital linear map Φ.

Proof. From $0 < mI \leq \mathcal{R}A \leq MI$, it follows that

$$(MI - \mathcal{R}A)(\frac{1}{m}I - \mathcal{R}^{-1}A) \geq 0,$$

which is equivalent to

$$\mathcal{R}A + Mm\mathcal{R}^{-1}(A) \leq (M+m)I.$$

Similarly

$$\mathcal{R}B + Mm\mathcal{R}^{-1}(B) \leq (M+m)I.$$

Since every operator mean is subadditive and monotone, we have

$$\begin{aligned}
\mathcal{R}A\sigma\mathcal{R}B + mM(\mathcal{R}^{-1}(A)\sigma\mathcal{R}^{-1}(B)) &\leq (\mathcal{R}A + Mm\mathcal{R}^{-1}(A))\sigma(\mathcal{R}B + Mm\mathcal{R}^{-1}(B)) \\
&\leq (M+m)I\sigma(M+m)I \\
&= (M+m)I
\end{aligned}$$

Thus, for a positive unital linear map Φ, we obtain

$$\Phi(\mathcal{R}A\sigma\mathcal{R}B) + mM\Phi(\mathcal{R}^{-1}(A)\sigma\mathcal{R}^{-1}(B)) \leq (M+m)I.$$

Let $\sigma^* \leq \sigma_1, \sigma_2 \leq \sigma$. Using (2.14) and Choi's inequality, we get

$$\begin{aligned}
\cos^2 \alpha \Phi \mathcal{R}(A\sigma_1 B) + mM\Phi^{-1}\mathcal{R}(A\sigma_2 B) &\leq \Phi(\mathcal{R}A\sigma_1\mathcal{R}B) + mM\Phi^{-1}(\mathcal{R}A\sigma_2\mathcal{R}B) \\
&\leq \Phi(\mathcal{R}A\sigma\mathcal{R}B) + mM\Phi^{-1}(\mathcal{R}A\sigma^*\mathcal{R}B) \\
&\leq \Phi(\mathcal{R}A\sigma\mathcal{R}B) + mM\Phi((\mathcal{R}A\sigma^*\mathcal{R}B)^{-1}) \\
&= \Phi(\mathcal{R}A\sigma\mathcal{R}B) + mM\Phi(\mathcal{R}^{-1}A\sigma\mathcal{R}^{-1}B) \\
&\leq (M+m)I,
\end{aligned}$$

and this completes the proof. □

Subsequently, the authors utilized the following lemmas to establish several important inequalities, which constitute the sectorial analogues of inequality (3).

Lemma 3.2. *[8] Let $A, B \geq 0$. Then*

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2.$$

Lemma 3.3. *[22] For each $A, B > 0$ and $p > 1$,*

$$\|A^p + B^p\| \leq \|(A+B)^p\|.$$

Theorem 3.4. *[21] Let $A, B \in \mathcal{S}_\alpha$ such that $0 < mI \le \mathcal{R}A, \mathcal{R}B \le MI$ and Φ be a positive unital linear map. If σ is an operator mean and σ_1, σ_2 are two operator means between σ and σ^*, then*

$$\mathcal{R}^p\Phi(A\sigma_1 B) \le \sec^{2p}\alpha\ K^p\mathcal{R}^p\Phi(A\sigma_2 B), \qquad (0 < p \le 2)$$
$$\mathcal{R}^p\Phi(A\sigma_1 B) \le \sec^{2p}\alpha 4^{p-2}K^p\mathcal{R}^p\Phi(A\sigma_2 B). \qquad (p \ge 2)$$

Proof. For the first inequality, by Löwner-Heinz inequality, it is enough to prove the inequality for $p = 2$.

Applying Lemma 3.2 and inequality (3.2), we have

$$\begin{aligned}
&\|\cos^2\alpha\Phi\mathcal{R}(A\sigma_1 B)\Phi^{-1}\mathcal{R}(A\sigma_2 B)\| \\
&\qquad \le \frac{1}{4mM}\|\cos^2\alpha\Phi\mathcal{R}(A\sigma_1 B) + mM\Phi^{-1}\mathcal{R}(A\sigma_2 B)\|^2 \\
&\qquad \le \frac{(M+m)^2}{4mM} = K,
\end{aligned}$$

which is equivalent to

$$\cos^4\alpha\Phi^2\mathcal{R}(A\sigma_1 B) \le K^2\Phi^2\mathcal{R}(A\sigma_2 B). \tag{3.3}$$

Now, if $p \ge 2$, by (2.14) and Lemma 3.3 and inequality ((3.2)), respectively

$$\begin{aligned}
&\|\cos^p\alpha\Phi^{\frac{p}{2}}\mathcal{R}(A\sigma_1 B)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}\mathcal{R}(A\sigma_2 B)\| \\
&\qquad \le \frac{1}{4}\left\|\cos^p\alpha\Phi^{\frac{p}{2}}\mathcal{R}(A\sigma_1 B) + M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}\mathcal{R}(A\sigma_2 B)\right\|^2 \\
&\qquad \le \frac{1}{4}\left\|\cos^2\alpha\Phi\mathcal{R}(A\sigma_1 B) + Mm\Phi^{-1}\mathcal{R}(A\sigma_2 B)\right\|^p \\
&\qquad \le \frac{1}{4}(M+m)^p.
\end{aligned}$$

Hence,

$$\|\cos^p\alpha\Phi^{\frac{p}{2}}\mathcal{R}(A\sigma_1 B)\Phi^{-\frac{p}{2}}\mathcal{R}(A\sigma_2 B)\| \le \left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^{\frac{p}{2}}.$$

This means that

$$\cos^{2p}\alpha\Phi^p\mathcal{R}(A\sigma_1 B) \le \left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p\Phi^p\mathcal{R}(A\sigma_2 B).$$

This completes the proof. □

The case $p = 2$ of this result was previously established in [3] with the constant $\sec^{12}\alpha$; however, the authors obtained a sharper bound with the improved constant $\sec^4\alpha$.

Moreover, by setting $p = 2$, $\sigma_1 = \nabla_t$, and $\sigma_2 = \sharp_t$ in Theorem 3.4, they recovered a refinement of certain inequalities due to [31] and [41]. By observing that $\mathcal{R}A\sigma_1\mathcal{R}B = \mathcal{R}(A\sigma_1 B)$ when $\sigma_1 = \nabla_t$—which eliminates the factor $\cos\alpha$—the authors proved the following theorem.

Theorem 3.5. *[21] Let $A, B \in \mathcal{S}_\alpha$ such that $0 < mI \leq \mathcal{R}A, \mathcal{R}B \leq MI$ and Φ be a positive unital linear map. If $!_t \leq \sigma \leq \nabla_t$, then*

$$\mathcal{R}^p\Phi(A\nabla_t B) \leq \alpha^p \mathcal{R}^p \Phi(A\sigma B),$$

where $\alpha = \max\{K, 4^{1-\frac{2}{p}}K\}$.

In addition, Lemma 1.1 was employed by the authors to prove another version of Theorem 3.4.

Theorem 3.6. *[21] Let $A, B \in \mathcal{S}_\alpha$ such that $0 < mI \leq \mathcal{R}A^{-1}, \mathcal{R}B^{-1} \leq MI$ and σ be an arbitrary mean. Then for every positive unital linear map Φ and $\sigma^* \leq \sigma_1, \sigma_2 \leq \sigma$,*

$$\Phi^p\mathcal{R}(A\sigma_1 B) \leq \alpha^p \sec^{4p}\alpha \Phi^p\mathcal{R}(A\sigma_2 B),$$

where $\alpha = \max\{K, 4^{1-\frac{2}{p}}K\}$.

Proof. It is enough to prove

$$Mm\Phi\mathcal{R}(A\sigma_1 B) + \Phi^{-1}\mathcal{R}(A\sigma_2 B) \leq \sec^2\alpha\ (M+m)I. \tag{3.4}$$

The rest of the proof is similar to Theorem 3.4.

As in the proof of Proposition 3.1,

$$\mathcal{R}A^{-1}\sigma\mathcal{R}B^{-1} + mM(\mathcal{R}^{-1}A^{-1}\sigma\mathcal{R}^{-1}B^{-1}) \leq (M+m)I.$$

From this inequality and Lemma 1.1, it follows that

$$\mathcal{R}A^{-1}\sigma\mathcal{R}B^{-1} + Mm(\mathcal{R}A\sigma\mathcal{R}B) \leq (M+m)I.$$

Inequality (2.14) leads to

$$\mathcal{R}(A\sigma_1 B) \leq \sec^2\alpha(\mathcal{R}A\sigma_1\mathcal{R}B) \leq \sec^2\alpha(\mathcal{R}A\sigma\mathcal{R}B). \tag{3.5}$$

In addition,

$$\begin{aligned}\Phi^{-1}\mathcal{R}(A\sigma_2 B) &\leq \Phi^{-1}(\mathcal{R}A\sigma_2\mathcal{R}B) \leq \Phi^{-1}(\mathcal{R}A\sigma^*\mathcal{R}B)\\ &\leq \Phi(\mathcal{R}A\sigma^*\mathcal{R}B)^{-1} = \Phi(\mathcal{R}^{-1}A\sigma\mathcal{R}^{-1}B)\\ &\leq \sec^2\alpha\Phi(\mathcal{R}A^{-1}\sigma\mathcal{R}B^{-1}).\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we get (3.1). □

A useful double inequality concerning operator monotone functions was established in [3]: for any $f \in \mathbf{m}$ and all $A \in \mathcal{S}_\alpha$ with $0 \leq \alpha < \frac{\pi}{2}$,

$$f(\mathcal{R}A) \leq \mathcal{R}(f(A)) \leq \sec^2\alpha\, f(\mathcal{R}A). \tag{3.7}$$

Subsequently, Khosravi et al. [21] established the following inequalities, which provide extensions of inequality (2.21).

Building on the framework of sectorial operators, the authors established several refined inequalities for operator monotone functions. Specifically, in [21, Theorem 2.13], they showed that for $A, B \in \mathcal{S}_\alpha$ with $0 < mI \leq \mathcal{R}A, \mathcal{R}B \leq MI$, any positive unital linear map Φ, any $f \in \mathbf{m}$, and any operator means σ_1, σ_2 lying between an arbitrary mean σ and its adjoint σ^*, the following chain of inequalities holds:

$$\mathcal{R}f\big(\Phi(A\sigma_1 B)\big) \leq \sec^2\alpha\,\mathcal{R}f\big(K\sec^2\alpha\,\Phi(A\sigma_2 B)\big) \leq K\sec^4\alpha\,\mathcal{R}f\big(\Phi(A\sigma_2 B)\big),$$

and the constant K can be omitted when $\sigma_1 \le \sigma_2$.

In [21, Theorem 2.14], under the same assumptions on A, B, Φ, and f, they proved that for any mean σ satisfying $!_t \le \sigma \le \nabla_t$,

$$\mathcal{R}f\big(\Phi(A\nabla_t B)\big) \le \sec^2\alpha\, \mathcal{R}f\big(K\,\Phi(A\sigma B)\big) \le K\sec^2\alpha\, \mathcal{R}f\big(\Phi(A\sigma B)\big).$$

Furthermore, in [21, Theorem 2.15], they presented the sectorial analogue of inequality (3.1): for $A, B \in \mathcal{S}_\alpha$ with $0 < mI \le \mathcal{R}A, \mathcal{R}B \le MI$, any positive unital linear map Φ, any $f \in \mathbf{m}$, and any means σ_1, σ_2 between ∇_t and $!_t$,

$$\mathcal{R}\big(f(\Phi(A))\sigma_1 f(\Phi(B))\big) \le K\sec^4\alpha\, \mathcal{R}f\big(\Phi(A\sigma_2 B)\big).$$

In the final part of this section, the authors developed determinant, singular value, and norm analogues of Theorems 3.4–3.6. Their approach relied on foundational inequalities for sectorial operators, notably the determinant estimate from [19, 24]:

$$\det(\mathcal{R}A) \le |\det(A)| \le \sec^n\alpha\, \det(\mathcal{R}A), \quad A \in \mathcal{S}_\alpha,$$

which served as a key tool for deriving determinant versions.

Accordingly, in [21, Theorem 3.2], they established determinant inequalities for sectorial means: under the condition $0 < mI \le \mathcal{R}A, \mathcal{R}B \le MI$, it held that

$$|\det(A\sigma_1 B)| \le \sec^{3n}\alpha\, K^n |\det(A\sigma_2 B)|,$$

and under the dual condition on the inverses, i.e., $0 < mI \le \mathcal{R}A^{-1}, \mathcal{R}B^{-1} \le MI$, the bound became $\sec^{5n}\alpha\, K^n$. Moreover, when $!_t \le \sigma \le \nabla_t$, they obtained

$$|\det(A\nabla_t B)| \le \sec^n\alpha\, K^n |\det(A\sigma B)|.$$

Similarly, using singular value estimates, [21, Theorem 3.4] provided the corresponding singular value inequalities: under the same assumptions on $\mathcal{R}A$ and $\mathcal{R}B$,

$$s_j(A\sigma_1 B) \le \sec^4\alpha\, K\, s_j(A\sigma_2 B),$$

while the inverse-bound condition yielded a $\sec^6\alpha$ factor. For means between $!_t$ and ∇_t, they further showed

$$s_j(A\nabla_t B) \le \sec^2\alpha\, K\, s_j(A\sigma B).$$

Parallel norm inequalities were presented in [21, Theorem 3.6]: for $0 < mI \le \mathcal{R}A, \mathcal{R}B \le MI$,

$$\|A\sigma_1 B\| \le \sec^3\alpha\, K\, \|A\sigma_2 B\|,$$

with the exponent rising to 5 under bounds on the inverses, and reducing to 1 in the case $!_t \le \sigma \le \nabla_t$.

Finally, in [21, Theorem 3.18], they proved additional determinant and norm estimates for an arbitrary mean $\sigma \le \nabla$:

$$|\det(A\sigma B)| \le \frac{\sec^{4n}\alpha}{2^n}\, |\det(I_n + A)|\, |\det(I_n + B)|,$$

and

$$\|A\sigma B\| \le \frac{\sec^4\alpha}{2}\, \|I_n + A\|\, \|I_n + B\|.$$

Bedrani et al. [3] proved that for $A \in \mathcal{S}_\alpha$, and any operator monotone function $f \in \mathbf{m}$, the following inequalities held:

$$f(\|\mathcal{R}A\|) \leq \|\mathcal{R}(f(A))\| \leq \sec^2 \alpha \, f(\|\mathcal{R}A\|). \tag{3.8}$$

Extending these ideas, the authors established in [21, Theorem 4.1] a numerical radius inequality for sectorial means: under the condition $0 < mI \leq \mathcal{R}A, \mathcal{R}B \leq MI$, and for any operator monotone function $f \in \mathbf{m}$ and means σ_1, σ_2 satisfying $\sigma^* \leq \sigma_1, \sigma_2 \leq \sigma$, one had

$$f(\omega(A\sigma_1 B)) \leq \sec^3 \alpha \, K \, \omega\big(f(A\sigma_2 B)\big).$$

Finally, in [21, Proposition 4.3], they presented a reverse-type numerical radius inequality, which refined [4, Theorem 3.5]: for $A, B \in \mathcal{S}_\alpha$ with $0 < mI \leq \mathcal{R}A, \mathcal{R}B \leq MI$, any positive unital linear map Φ, any $f \in \mathbf{m}$, and any means σ_1, σ_2 between ∇_t and $!_t$,

$$\omega\big(f(\Phi(A))\sigma_1 f(\Phi(B))\big) \leq K \sec^5 \alpha \, \omega\big(f(\Phi(A\sigma_2 B))\big).$$

4. Hadamard and Kronecker Products for Sector Matrices

Among the most useful tools in the study of positive semidefinite matrices are the Kronecker and Hadamard products.

Let $A = [a_{ij}] \in \mathbb{M}_{m\times n}$ and $B = [b_{ij}] \in \mathbb{M}_{p\times q}$. The *Kronecker (or tensor) product* of A and B, denoted by $A \otimes B$, is the matrix in $\mathbb{M}_{mp\times nq}$ defined by

$$A \otimes B = [a_{ij}B] = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

It is well known (see, for example, [19]) that the cone of positive semidefinite matrices is closed under the Kronecker product; that is, if $A \geq 0$ and $B \geq 0$, then their Kronecker product satisfies $A \otimes B \geq 0$.

The *Hadamard product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathbb{M}_{m\times n}$, denoted by $A \circ B$, is defined entrywise by

$$A \circ B = [a_{ij}b_{ij}].$$

This operation, also known as the *Schur product*, preserves positivity in the sense that if $A \geq 0$ and $B \geq 0$ in $\mathbb{M}_n$, then $A \circ B \geq 0$, a result famously known as the Schur product theorem.

A fundamental property of the Kronecker product is its compatibility with positivity: whenever A and B are positive semidefinite matrices (i.e., $A \geq 0$ and $B \geq 0$), their Kronecker product $A \otimes B$ is also positive semidefinite (see, e.g., [19]).

It is well known (see, e.g., [33]) that the Hadamard product admits a natural representation via the Kronecker product. Specifically, let $\{e_j\}_{j=1}^n$ be an orthonormal basis of $\mathbb{C}^n$, and let $V : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ be the isometry defined by $Ve_j = e_j \otimes e_j$ for each j (so that $V^*V = I$). Then, with respect to the basis $\{e_j\}_{j=1}^n$, the Hadamard product of two matrices $A, B \in \mathbb{M}_n$ is given by

$$A \circ B = V^*(A \otimes B)V. \tag{4.1}$$

From identity (4.1), it follows that the Hadamard product $A \circ B$ is a principal submatrix of the Kronecker product $A \otimes B$. Consequently, one has $A \otimes B \geq 0$ if and only if $A \circ B \geq 0$.

Moreover, the same representation immediately yields

$$\mathcal{R}(A \circ B) = V^* \mathcal{R}(A \otimes B) V,$$

and therefore $\mathcal{R}(A \otimes B) \geq 0$ if and only if $\mathcal{R}(A \circ B) \geq 0$.

The Kronecker product possesses numerous useful and elegant properties (see, e.g., [19, Chapter 4]). Among these, one of the most fundamental is the *mixed-product property*, which asserts that

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \tag{4.2}$$

whenever the matrix products AC and BD are defined that is, when the dimensions of A, B, C, D are compatible.

Moreover, for any positive semidefinite matrices $A, B \in \mathbb{M}_n^+$ and any real exponent $t \in \mathbb{R}$, the Kronecker product commutes with matrix powers:

$$(A \otimes B)^t = A^t \otimes B^t. \tag{4.3}$$

It is well known that the set of $m \times n$ complex matrices forms an abelian (i.e., commutative) group under the Hadamard product. Moreover, the Hadamard product is bilinear over $\mathbb{C}$. Specifically, for any scalars $\alpha, \beta \in \mathbb{C}$ and any $m \times n$ matrices A, B, and C, one has

$$C \circ (A + B) = C \circ A + C \circ B,$$

and

$$\alpha(A \circ B) = (\alpha A) \circ B = A \circ (\alpha B).$$

As an immediate consequence,

$$(\alpha A) \circ (\beta B) = \alpha\beta\, (A \circ B). \tag{4.4}$$

Recent advances concerning sectorial matrices and their applications were reported in [3–6].

The Hadamard product distributes over matrix addition. Consequently, for any $A, B \in \mathbb{M}_n$ yields

$$\begin{aligned} A \circ B &= (\mathcal{R}A + i\,\mathcal{I}A) \circ (\mathcal{R}B + i\,\mathcal{I}B) \\ &= (\mathcal{R}A \circ \mathcal{R}B - \mathcal{I}A \circ \mathcal{I}B) + i\,(\mathcal{R}A \circ \mathcal{I}B + \mathcal{I}A \circ \mathcal{R}B). \end{aligned}$$

Taking real parts gives the identity

$$\mathcal{R}(A \circ B) = \mathcal{R}A \circ \mathcal{R}B - \mathcal{I}A \circ \mathcal{I}B. \tag{4.5}$$

An analogous computation for the Kronecker product leads to

$$\mathcal{R}(A \otimes B) = \mathcal{R}A \otimes \mathcal{R}B - \mathcal{I}A \otimes \mathcal{I}B.$$

Recall that the Hadamard product preserves positivity: if $A, B \in \mathbb{M}_n^+$, then $A \circ B \geq 0$ (see, e.g., [19, Chapter 4]). Moreover, it is monotone with respect to the Löwner order; specifically, if $A_1 \geq A_2 \geq 0$ and $B_1 \geq B_2 \geq 0$, then

$$A_1 \circ B_1 \geq A_2 \circ B_2 \geq 0. \tag{4.6}$$

It should be noted, however, that the Hadamard product of two accretive matrices need not be accretive. For instance, taking $A = B = 1 + i$ (viewed as 1×1 matrices), one obtains $A \circ B = (1+i)^2 = 2i$, whose real part is zero; hence $A \circ B$ is not accretive.

From the identity (4.5), it follows that if $\mathcal{I}A \circ \mathcal{I}B \leq 0$, then $\mathcal{R}(A \circ B) \geq \mathcal{R}A \circ \mathcal{R}B$. In particular, when $\mathcal{R}A \geq 0$, $\mathcal{R}B \geq 0$, and $\mathcal{I}A \circ \mathcal{I}B \leq 0$ (for example, if $\mathcal{I}A \geq 0$ and $\mathcal{I}B \leq 0$), the Hadamard product $A \circ B$ is accretive. However, this condition is sufficient but not necessary.

Since $\mathcal{R}(A \circ B)$ is Hermitian, it is eligible for inclusion in matrix inequalities even when it is not positive semidefinite. The following result follows directly from the identity (4.5).

Lemma 4.1. *[11] Let $A, B \in \mathbb{M}_n$ be accretive matrices. Then*

- *If $\mathcal{I}A \circ \mathcal{I}B \geq 0$, then*
$$\mathcal{R}(A \circ B) \leq \mathcal{R}A \circ \mathcal{R}B. \tag{4.7}$$
- *If $\mathcal{I}A \circ \mathcal{I}B \leq 0$, then*
$$\mathcal{R}A \circ \mathcal{R}B \leq \mathcal{R}(A \circ B). \tag{4.8}$$

Attention is now directed toward Hadamard product inequalities for accretive matrices that involve the powers t and $1-t$, with $t \in (-1, 0) \cup (1, 2)$.

In [10, Theorem 2], Chansangiam et al. established the following inequality: for any $A, B, C, D \in \mathbb{M}_n^+$, scalars $\alpha, \beta > 0$, and $t \in [0, 1]$, one has

$$\big(\alpha A + \beta B\big)^t \circ \big(\alpha C + \beta D\big)^{1-t} \;\geq\; \alpha\big(A^t \circ C^{1-t}\big) + \beta\big(B^t \circ D^{1-t}\big). \tag{4.9}$$

Lemma 4.2. *[11] Let $A, B \in \mathcal{S}_\alpha$. If $t \in [0, 1]$, then*
$$\cos^{2t} \alpha \mathcal{R}A^t \leq \mathcal{R}^t A \leq \mathcal{R}A^t \tag{4.10}$$
and if $t \in [-1, 0]$, then
$$\mathcal{R}A^t \leq \mathcal{R}^t A \leq \cos^{2t} \alpha \mathcal{R}A^t. \tag{4.11}$$

Sheikhhosseini et al. extended inequality (4.9) in [35] using the above lemma.

Theorem 4.3. *[35] Let $A, B, C, D \in \mathcal{S}_\alpha$ be such that $\mathcal{I}A^t \circ \mathcal{I}C^{1-t} \geq 0$ and $\mathcal{I}B^t \circ \mathcal{I}D^{1-t} \geq 0$, where $0 < t < 1$. If $\alpha, \beta > 0$, then*
$$\mathcal{R}\left(\alpha(A^t \circ C^{1-t}) + \beta(B^t \circ D^{1-t})\right) \leq \sec^2 \alpha \left(\mathcal{R}^t\left(\alpha A + \beta B\right) \circ \mathcal{R}^{1-t}\left(\alpha C + \beta D\right)\right).$$

A map $\Phi : \mathbb{M}_n \times \mathbb{M}_n \longrightarrow \mathbb{M}_m$ is said to be jointly convex if
$$\Phi(\lambda A + (1-\lambda)B, \lambda C + (1-\lambda)D) \leq \lambda\Phi(A, C) + (1-\lambda)\Phi(B, D)$$
for all $\lambda \in [0, 1]$ and for every $A, B \in \mathbb{M}_n$. They employed the following lemma and proposition to obtain the reverse of inequality (4.9) for $t \in (-1, 0) \cup (1, 2)$.

Lemma 4.4. *[9] If $A, B \in \mathbb{M}_n$ are such that $A \geq 0$, then the map $(A, B) \to BA^{-1}B$ is jointly convex.*

Proposition 4.5. *[35] If $A, B, C, D \in \mathbb{M}_n^+$ and $\alpha, \beta > 0$, then*
$$(\alpha A + \beta B)^t \otimes (\alpha C + \beta D)^{1-t} \leq \alpha\left(A^t \otimes C^{1-t}\right) + \beta\left(B^t \otimes D^{1-t}\right), \tag{4.12}$$
where $t \in (-1, 0) \cup (1, 2)$.

Proof. First, we prove the assertion for $t \in (-1,0)$. For $t \in (1,2)$, it is sufficient to set $1-t$ instead of t, A instead of C and B instead of D.
For $t \in (-1,0)$, by [9, p. 23], we have

$$x^t = \int_0^\infty (s+x)^{-1} d\mu(s),$$

where μ is a positive measure on $(0,\infty)$. Therefore

$$(A \otimes B^{-1})^t = \int_0^\infty (sI + (A \otimes B^{-1}))^{-1} d\mu(s). \tag{4.13}$$

So, by (4.2) and (4.3), we get

$$\begin{aligned}
&A^t \otimes B^{1-t} = (A^t I) \otimes B^{-t} B) \\
&= (A^t \otimes B^{-t})(I \otimes B) = (A \otimes B^{-1})^t (I \otimes B) \\
&= \int_0^\infty (I \otimes B) \left(sI \otimes I + (A \otimes B^{-1})\right)^{-1} d\mu(s) \qquad \text{(by (4.13))} \\
&= \int_0^\infty (I \otimes B) \left(sI \otimes I + (A \otimes B^{-1})\right)^{-1} (I \otimes B)^{-1} (I \otimes B) d\mu(s) \\
&= \int_0^\infty (I \otimes B) \left((sI \otimes I)(I \otimes B) + (A \otimes B^{-1})(I \otimes B)\right)^{-1} (I \otimes B) d\mu(s) \\
&= \int_0^\infty (I \otimes B) \left((sI \otimes B) + (A \otimes I)\right)^{-1} (I \otimes B) d\mu(s).
\end{aligned}$$

Since $(sI \otimes B) + (A \otimes I)$ is positive definite, the map $(I \otimes B) \left((sI \otimes B) + (A \otimes I)\right)^{-1} (I \otimes B)$ by Lemma 4.4, is jointly convex. Hence, the integrand is jointly convex, and so is $A^t \otimes B^{1-t}$. This means that for any $A, B, C, D \in \mathbb{M}_n^+$ and scalar $0 < \epsilon < 1$,

$$(\epsilon A + (1-\epsilon)B)^t \otimes (\epsilon C + (1-\epsilon)D)^{1-t} \leq \epsilon (A^t \otimes C^{1-t}) + (1-\epsilon)(B^t \otimes D^{1-t}).$$

Since $0 < \alpha/(\alpha+\beta) < 1$, by setting $\epsilon = \alpha/(\alpha+\beta)$, we get (4.12). $\square$

The following theorem established inequality (4.12) for the Hadamard product of positive definite matrices.

Theorem 4.6. *[35] Let $A, B, C, D \in \mathbb{M}_n^+$ and $\alpha, \beta > 0$. Then*

$$(\alpha A + \beta B)^t \circ (\alpha C + \beta D)^{1-t} \leq \alpha \left(A^t \circ C^{1-t}\right) + \beta \left(B^t \circ D^{1-t}\right), \tag{4.14}$$

where $t \in (-1,0) \cup (1,2)$.

Proof. Define $\Phi : \mathbb{M}_n^+ \times \mathbb{M}_n^+ \to \mathbb{M}_{n^2}^+$ by $\Phi(A,B) = A^t \otimes B^{1-t}$. Recall that the Hadamard product of matrices is a principal submatrix of the Kronecker product. Consequently, there exists a positive unital linear map $\Psi : \mathbb{M}_{n^2}^+ \to \mathbb{M}_n^+$ such that $\Psi(A \otimes B) = A \circ B$. Hence,

$$(\Psi \circ \Phi)(A,B) = \Psi(\Phi(A,B)) = \Psi(A^t \otimes B^{1-t}) = A^t \circ B^{1-t}.$$

Since by Proposition 4.5, Φ is jointly convex and Ψ is positive and linear, the composition $\Psi \circ \Phi$ is also jointly convex. This means that for any $A, B, C, D \in \mathbb{M}_n^+$ and any scalar

$0 < \epsilon < 1$,

$$(\epsilon A + (1-\epsilon)B)^t \circ (\epsilon C + (1-\epsilon)D)^{1-t} \leq \epsilon(A^t \circ C^{1-t}) + (1-\epsilon)(B^t \circ D^{1-t}).$$

Since $0 < \alpha/(\alpha+\beta) < 1$, by setting $\epsilon = \alpha/(\alpha+\beta)$, we get (4.14). □

The authors extended inequality (4.14) in the following theorem.

Theorem 4.7. *[35] If $A, B, C, D \in \mathcal{S}_\alpha$ are such that $\mathcal{I}A^t \circ \mathcal{I}C^{1-t} \leq 0$ and $\mathcal{I}B^t \circ \mathcal{I}D^{1-t} \leq 0$, then*

$$\mathcal{R}^t(\alpha A + \beta B) \circ \mathcal{R}^{1-t}(\alpha C + \beta D) \leq \sec^2 \alpha \mathcal{R}\left(\alpha\left(A^t \circ C^{1-t}\right) + \beta\left(B^t \circ D^{1-t}\right)\right),$$

where $t \in (-1, 0) \cup (1, 2)$ and $\alpha, \beta > 0$.

Sheikhhosseini et al. subsequently established several new inequalities for accretive matrices and provided accretive analogues of well-known results that hold for positive definite matrices. By applying the following lemma, they also obtained a number of Hadamard-type inequalities involving matrix monotone functions.

Lemma 4.8. *[33, Theorem 6.3] Let $A, B \geq 0$ and Φ be a positive unital linear map. If f is a super multiplicative matrix concave function on $(0, \infty)$, then*

$$f(\Phi(A \circ B)) \geq \Phi(f(A) \circ f(B)). \tag{4.15}$$

The authors extended inequality (4.15) in the following theorem.

Theorem 4.9. *[35] Let $A, B \in \mathcal{S}_\alpha$ be such that $\mathcal{I}A \circ \mathcal{I}B \leq 0$ and Φ be a positive unital linear map. If f is a super multiplicative matrix concave function on $(0, \infty)$, then*

$$\Phi(\mathcal{R}f(A) \circ \mathcal{R}f(B)) \leq \sec^4 \alpha \; \mathcal{R}f(\Phi(A \circ B)).$$

In the study of matrix means, it is customary to compare different means associated with distinct matrix monotone functions. Theorem 4.10 provided such a comparison for sectorial matrices.

Theorem 4.10. *[35] Let $A, B \in \mathcal{S}_\alpha$ be such that $0 < mI \leq \mathcal{R}A, \mathcal{R}B \leq MI$ for some scalars $m < M$ and assume $\mathcal{I}A \circ \mathcal{I}B \leq 0$. If σ_1 and σ_2 are two matrix means between σ, σ^* for some matrix mean σ, then for every positive unital linear map Φ,*

$$\left\|\Phi^{\frac{1}{2}}(\mathcal{R}(A\sigma_1 B)) \circ \Phi^{-\frac{1}{2}}(\mathcal{R}(A\sigma_2 B))\right\| \leq \frac{M+m}{2\sqrt{mM}} \sec \alpha.$$

Zhang [43, p. 204] established a relationship between the inverse of the Hadamard product and the Hadamard product of inverses for positive definite matrices, and Sheikhhosseini et al. later extended these results to sectorial matrices.

Lemma 4.11. *[43, p. 204] Let $A \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_{n\times m}$ be such that $X^*X = I_m$. Then*

$$(X^*AX)^{-1} \leq X^*A^{-1}X \leq \frac{(\lambda+\mu)^2}{4\lambda\mu}(X^*AX)^{-1}, \tag{4.16}$$

where λ is the largest and μ is the smallest eigenvalue of A.

Specially, if $A, B \in \mathbb{M}_n^+$, by choosing appropriate X and replacing A with $A \otimes B$, we get

$$(A \circ B)^{-1} \leq A^{-1} \circ B^{-1} \leq \frac{(\lambda+\mu)^2}{4\lambda\mu}(A \circ B)^{-1}, \tag{4.17}$$

where λ is the largest and μ is the smallest eigenvalue of $A \otimes B$.

Theorem 4.12. *[35] Let $A, B \in \mathcal{S}_\alpha$ be such that $\mathcal{I}A \circ \mathcal{I}B \leq 0$. Then*

$$\cos^4 \alpha \mathcal{R}(A \circ B)^{-1} \leq \mathcal{R}A^{-1} \circ \mathcal{R}B^{-1}.$$

The following theorem is a reverse version of Theorem 4.12.

Theorem 4.13. *[35] Let $A, B, A \circ B \in \mathcal{S}_\alpha$ be such that either*

1) *$\mathcal{I}A^{-1} \circ \mathcal{I}B^{-1} \leq 0$ and $mI \leq \mathcal{R}(A \otimes B) \leq MI$ for some scalars $0 < m \leq M$,* or
2) *$\mathcal{I}A \circ \mathcal{I}B \geq 0$ and $mI \leq \mathcal{R}A \otimes \mathcal{R}B \leq MI$ for some scalars $0 < m \leq M$.*

Then

$$\mathcal{R}A^{-1} \circ \mathcal{R}B^{-1} \leq \sec^2 \alpha \frac{(M+m)^2}{4mM} \mathcal{R}(A \circ B)^{-1}.$$

Proof. For the first condition, in view of (4.1) and (4.3), there exists an isometry X of appropriate size such that $A^{-1} \circ B^{-1} = X^*(A \otimes B)^{-1}X$. Then using (4.8), (4.11) and (4.16) respectively, we have

$$\begin{aligned}
\mathcal{R}A^{-1} \circ \mathcal{R}B^{-1} &\leq \mathcal{R}(A^{-1} \circ B^{-1}) = \mathcal{R}(X^*(A \otimes B)^{-1}X) \\
&= X^*\mathcal{R}(A \otimes B)^{-1}X \leq X^*\mathcal{R}^{-1}(A \otimes B)X \\
&\leq \frac{(M+m)^2}{4Mm}(X^*\mathcal{R}(A \otimes B)X)^{-1} \\
&= \frac{(M+m)^2}{4Mm}\mathcal{R}^{-1}(X^*(A \otimes B)X) \\
&= \frac{(M+m)^2}{4Mm}\mathcal{R}^{-1}(A \circ B) \\
&\leq \sec^2 \alpha \frac{(M+m)^2}{4Mm}\mathcal{R}(A \circ B)^{-1}.
\end{aligned}$$

To prove the result under the second condition, applying (4.11), (4.17) and (4.7) sequentially, we get

$$\begin{aligned}
\mathcal{R}A^{-1} \circ \mathcal{R}B^{-1} &\leq \mathcal{R}^{-1}A \circ \mathcal{R}^{-1}B \\
&\leq \frac{(M+m)^2}{4Mm}(\mathcal{R}A \circ \mathcal{R}B)^{-1} \\
&\leq \frac{(M+m)^2}{4Mm}\mathcal{R}^{-1}(A \circ B) \\
&\leq \sec^2 \alpha \frac{(M+m)^2}{4Mm}\mathcal{R}(A \circ B)^{-1}.
\end{aligned}$$

So, the result holds. □

5. The $\mathcal{C}_{M,m}$-Transform and Matrix Inequalities for Sectorial Operators

Dragomir [13] defined the transform $\mathcal{C}_{M,m} : \mathbb{M}_n \to \mathbb{M}_n$ by

$$\mathcal{C}_{M,m}(A) = (MI - A^*)(A - mI),$$

where $M > m > 0$ are two real numbers and I is the identity matrix in $\mathbb{M}_n$.

In [18], the authors proved that, if $\mathcal{C}_{M,m}(A)$ is accretive, then A is accretive too. But if A is accretive, then $\mathcal{C}_{M,m}(A)$ is not accretive. For example

$$A=\begin{bmatrix}1 & 0\\ \frac{1}{2} & 1\end{bmatrix}\Rightarrow \mathcal{R}A=\frac{A+A^*}{2}=\begin{bmatrix}1 & \frac{1}{4}\\ \frac{1}{4} & 1\end{bmatrix}\geq 0$$

and

$$\begin{aligned}\mathcal{C}_{M,m}(A)&=(MI-A^*)\,(A-mI)\\&=\begin{bmatrix}M-1 & -\frac{1}{2}\\ 0 & M-1\end{bmatrix}\begin{bmatrix}1-m & 0\\ \frac{1}{2} & 1-m\end{bmatrix}\\&=\begin{bmatrix}(M-1)(1-m)+\frac{1}{4} & \frac{m-1}{2}\\ \frac{M-1}{2} & (M-1)(1-m)\end{bmatrix},\end{aligned}$$

therefore

$$\mathcal{R}\mathcal{C}_{M,m}(A)=\begin{bmatrix}(M-1)(1-m)-\frac{1}{4} & \frac{m+M}{4}-\frac{1}{2}\\ \frac{m+M}{4}-\frac{1}{2} & (M-1)(1-m)\end{bmatrix}$$

for $m=2$ and $M=3$, we have

$$\mathcal{R}\mathcal{C}_{M,m}(A)=\begin{bmatrix}-\frac{9}{4} & \frac{3}{4}\\ \frac{3}{4} & -2\end{bmatrix}.$$

It is clear that $\mathcal{C}_{M,m}(A)$ is not accretive.

Some of the main and interesting properties of the transform $\mathcal{C}_{M,m}(\cdot)$ were presented in [13, 18, 32].

Gümüş et al. [18] proved that if $A\in\mathbb{M}_n$ and $\mathcal{C}_{M,m}(A)$ is accretive, then

$$|A|\leq\frac{M+m}{2\sqrt{Mm}}\mathcal{R}A,$$

and

$$\mathcal{R}A\leq\frac{M+m}{2\sqrt{Mm}}|A|.$$

Using these inequalities, Malekinejad et al. [28] established the following results:

Theorem 5.1. *[28] Let $A,B\in\mathbb{M}_n$, $\mathcal{C}_{M,m}(A)$ and $\mathcal{C}_{M,m}(B)$ be accretive. Then for every two matrix means $\sigma_1\leq\sigma_2$, we have*

$$|A|\sigma_1|B|\leq\frac{M+m}{2\sqrt{Mm}}\mathcal{R}(A\sigma_2 B).$$

By selecting $B=iA^*$ in Theorem 5.1, the authors obtained the following consequence, which extends [18, Theorem 2.3].

Corollary 5.2. *[28] Let $A\in\mathbb{M}_n$, $\mathcal{C}_{M,m}(A)$ and $\mathcal{C}_{M,m}(iA^*)$ be accretive. Then for every two matrix means $\sigma_1\leq\sigma_2$, we have*

$$|A|\sigma_1|A^*|\leq\frac{M+m}{2\sqrt{Mm}}\mathcal{R}A\sigma_2\mathcal{I}A.$$

Complementary to Theorem 5.1, the following results provide further upper bounds for $\mathcal{R}(A\sigma_1 B)$ under alternative accretivity assumptions namely, on the inverses A^{-1}, B^{-1} or on the absolute values $|A|,|B|$. Both are due to Malekinejad et al. [28].

Theorem 5.3. *[28] Let $A, B \in \mathbb{M}_n$, $\mathcal{C}_{M,m}(A^{-1})$ and $\mathcal{C}_{M,m}(B^{-1})$ be accretive. Then for every matrix means $\sigma_1 \leq \sigma_2$,*

$$\mathcal{R}(A\sigma_1 B) \leq \frac{M+m}{2\sqrt{Mm}}(|A^*|\sigma_2|B^*|).$$

Theorem 5.4. *[28] Let $A, B \in \mathcal{S}_\alpha$, $\mathcal{C}_{M,m}(|A|)$ and $\mathcal{C}_{M,m}(|B|)$ be accretive. Then for every two matrix means $\sigma_1 \leq \sigma_2$, we have*

$$\mathcal{R}(A\sigma_1 B) \leq \frac{M+m}{2\sqrt{Mm}} \sec^2 \alpha(|A|\sigma_2|B|).$$

Combining Theorem 5.1 and Theorem 5.3, the authors deduced the following corollary.

Corollary 5.5. *[28] Let $\mathcal{C}_{M,m}(A), \mathcal{C}_{M,m}(B), \mathcal{C}_{M,m}(A^{-1}), \mathcal{C}_{M,m}(B^{-1})$ be accretive. Then*

$$\frac{2\sqrt{Mm}}{M+m}\left(|A|\sigma|B|\right) \leq \mathcal{R}(A\sigma B) \leq \frac{M+m}{2\sqrt{Mm}}\left(|A^*|\sigma|B^*|\right).$$

From Theorem 5.1 together with Theorem 5.4, the following corollary follows immediately.

Corollary 5.6. *[28] Let $A, B \in \mathcal{S}_\alpha$, and $\mathcal{C}_{M,m}(A)$, $\mathcal{C}_{M,m}(|A|)$, $\mathcal{C}_{M,m}(B)$ and $\mathcal{C}_{M,m}(|B|)$ be accretive. Then for every matrix mean σ,*

$$\frac{2\sqrt{Mm}}{M+m}\left(|A|\sigma|B|\right) \leq \mathcal{R}(A\sigma B) \leq \frac{M+m}{2\sqrt{Mm}} \sec^2 \alpha \left(|A|\sigma|B|\right).$$

Definition 5.7. *Let $h > 0$. A generalized Kantorovich constant $K(h,p)$ (briefly $K(p)$) is defined by*

$$K(h,p) = \frac{(h^p - h)}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p-1}{h^p-h}\right) \tag{5.1}$$

where $p \in \mathbb{R}$.

If $h = \frac{M}{m}$, then

$$K(h,p) = \frac{(mM^p - Mm^p)}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p-m^p}{mM^p-Mm^p}\right)^p,$$

which is denoted by $K(m,M,p)$.

It is worth noting that $K(m,M,p)$ extends the classical Kantorovich constant $\dfrac{(M+m)^2}{4Mm}$. Indeed, one has $K(m,M,2) = K(m,M,-1) = \frac{(M+m)^2}{4Mm}$. For further properties and detailed analysis of the generalized Kantorovich constant $K(h,p)$, the reader is referred to [33].

The famous Hölder–McCarthy inequality says:

Lemma 5.8. *([33, Theorem 1.4]) Let $A \geq 0$. Then*
(i) $x^*A^tx \geq (x^*Ax)^t$ for all $r > 1$ and every unit vector $x \in \mathbb{C}^n$.
(ii) $(x^*A^tx) \leq (x^*Ax)^t$ for all $0 < r < 1$ and every unit vector $x \in \mathbb{C}^n$.
(iii) If A is invertible, then $(x^*A^tx) \geq (x^*Ax)^t$ for all $r < 0$ and every unit vector $x \in \mathbb{C}^n$.

Malekinejad et al. [28] derived the converse of the Hölder–McCarthy inequality by employing the generalized Kantorovich constant.

Lemma 5.9. *([33, Theorem 2.53]) Let $MI \geq A \geq mI$ for some scalars $0 < m < M$. Then*

$$(x^*Ax)^p \leq (x^*A^px) \leq K(m, M, p)(x^*Ax)^p \quad \text{for } p \notin [0, 1].$$

Theorem 5.10. *[28] Let $A \in \mathbb{M}_n$ such that $0 < mI \leq |A| \leq MI$. If $p \notin [0, 2)$, then*

$$\mathcal{R}A \leq K_p^{\frac{1}{p}}|A|. \tag{5.2}$$

where $K_p = K(m, M, p)$.

Corollary 5.11. *[28] Let $A \in \mathbb{M}_n$ such that $0 < mI \leq |A^*| \leq MI$. Then for $p \notin [0, 2)$,*

$$\mathcal{R}A \leq K_p^{\frac{1}{p}}|A^*|. \tag{5.3}$$

Corollary 5.12. *[28] If $A \in \mathbb{M}_n$ is invertible and $0 < mI \leq |A^{-1}| \leq MI$, then for $p \notin [0, 2)$,*

$$\mathcal{I}A^{-1} \leq K_p^{\frac{1}{p}}|A^{-1}|. \tag{5.4}$$

Building on the converse Hölder–McCarthy inequality and the properties of the generalized Kantorovich constant, Malekinejad et al. [28] further extended the above estimates to sectorial matrices. The following theorem provides a refined upper bound for $\mathcal{R}(A\sigma_1B)$ in terms of $|A|\sigma_2|B|$, incorporating both the sector angle α and the constant K_p.

Theorem 5.13. *[28] Let $A, B \in \mathcal{S}_\alpha$ such that $0 < mI \leq |A|, |B| \leq MI$. If $p \notin [0, 2)$, then for every two matrix means $\sigma_1 \leq \sigma_2$, we have*

$$\mathcal{R}(A\sigma_1B) \leq K_p^{\frac{1}{p}} \sec^2\alpha \left(|A|\sigma_2|B|\right). \tag{5.5}$$

where $K_p = K(m, M, p) = \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p}\right)^p$.

6. Extending Operator Monotone Function Inequalities to Sectorial Matrices

In the study of matrix and operator monotone functions, Choi's inequality plays a fundamental role. For a unital positive linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$, a positive definite matrix $A \in \mathbb{M}_n$, and an operator monotone function $f : (0, \infty) \to (0, \infty)$, Choi's inequality takes the following form:

$$\Phi(f(A)) \leq f(\Phi(A)), \tag{6.1}$$

Ghazanfari and Malekinejad [16] focused on generalizing the theory of operator monotone functions, extending the boundaries of this concept towards accretive matrices. The authors provided a new analytical framework for studying the behavior of these functions under conditions beyond strictly positive matrices by proving an equivalent integral formulation for $f(A)$ and by presenting bounds on the Thompson distance within this space.

This section is dedicated to examining the extensions achieved by Ghazanfari and Malekinejad [16], specifically their generalization of most of the results from equations (3.7) and (6.1) from the setting of positive matrix monotone functions to the domain of real matrix monotone functions.

Let A be an accretive operator on $\mathcal{B}(\mathcal{H})$, then

$$\mathcal{R}(Sp(A)) \subseteq \mathcal{R}(\overline{W(A)}) \subseteq \overline{\mathcal{R}(W(A))} = \overline{W(\mathcal{R}(A))}.$$

It is trivial that if $\lambda, \mu \geq 0$ and $v \in \mathbb{R}$, then $\frac{\lambda}{\sqrt{(\lambda+\mu)^2+v^2}} \leq 1$. Therefore, if A is an accretive operator on $\mathcal{B}(\mathcal{H})$, then

$$\begin{aligned} \|\lambda(\lambda + A)^{-1}\| &= \sup \left\{ \left| \frac{\lambda}{\lambda + \omega} \right| : \omega \in Sp(A) \right\} \\ &= \sup \left\{ \frac{\lambda}{\sqrt{(\lambda+\mu)^2 + v^2}} : \omega = \mu + iv \in Sp(A) \right\} \leq 1. \end{aligned}$$

Theorem 6.1. *[16, Theorem 2.1] Let $f : (0, \infty) \to \mathbb{R}$ be an operator monotone function and $A \in \mathbb{M}_n$ be an accretive matrix. Then*

$$f(A) = a + bA + \int_0^\infty \lambda A(\lambda + A)^{-1} d\mu(\lambda), \tag{6.2}$$

where $b \geq 0$, $a \in \mathbb{R}$ and μ is a positive measure on the closed positive half-line $[0, \infty)$ with $\int \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty$.

Proof. It is known that every operator monotone function f on $(0, \infty)$ has a special integral representation as follows [7, (V.53)]:

$$f(t) = a + bt + \int_0^\infty \frac{\lambda t}{\lambda + t} d\mu(\lambda), \tag{6.3}$$

where a, b are real numbers, $b \geq 0$, and μ is a positive measure with $\int \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty$.

This representation shows that f has the following analytic continuation on complex plane except on $(-\infty, 0]$,

$$f(z) = a + bz + \int_0^\infty \frac{\lambda z}{\lambda + z} d\mu(\lambda). \tag{6.4}$$

Since $\mathcal{I}\left(\frac{z}{\lambda+z}\right) = \frac{\lambda^2 \mathcal{I}z}{|\lambda+z|^2}$, the function defined above in (6.4) maps the upper half-plane into itself. We have

$$\lambda \leq \lambda + \mathcal{R}z \leq |\lambda + z| \Rightarrow \left| \frac{(\lambda+1)z}{\lambda + z} \right| \leq \frac{(\lambda+1)(|\mathcal{R}z| + |\mathcal{I}z|)}{\lambda + \mathcal{R}z} \leq M < \infty.$$

Therefore,

$$\int_0^\infty \frac{\lambda z}{\lambda + z} d\mu(\lambda) = \int_0^\infty \frac{(\lambda+1)z}{\lambda + z} \frac{\lambda}{\lambda + 1} d\mu(\lambda) < \infty.$$

This implies that if D is a diagonal matrix, then

$$f(D) = a + bD + \int_0^\infty \lambda D(\lambda + D)^{-1} d\mu(\lambda).$$

If A is diagonalizable, then $A = V^{-1}DV$ and

$$f(A) = f(V^{-1}DV) = V^{-1}f(D)V = a + bA + \int_0^\infty \lambda A(\lambda + A)^{-1} d\mu(\lambda).$$

Now, for a general accretive matrix A, let (D_n) be a sequence of diagonalizable matrices converging to A (by density of diagonalizable matrices), in the operator norm. From $D_n \to A$, we get

$$\langle Ax, x\rangle = \lim_n \langle D_n x, x\rangle$$

and

$$\mathcal{R}\langle Ax, x\rangle = \lim_n \mathcal{R}\langle D_n x, x\rangle.$$

Therefore without losing generality, we can assume that the sequence (D_n) is accretive diagonalizable.

Since $\|(\lambda(\lambda + A)^{-1})\| \leq 1$, the following inequalities hold

$$\begin{aligned}
&\|\lambda D_n(\lambda + D_n)^{-1} - \lambda A(\lambda + A)^{-1}\| \\
&= \|\lambda(\lambda + D_n)^{-1}\big[D_n(\lambda + A) - (\lambda + D_n)A\big](\lambda + A)^{-1}\| \\
&= \|\lambda(\lambda + D_n)^{-1}(D_n - A)\lambda(\lambda + A)^{-1}\| \leq \|\lambda(\lambda + D_n)^{-1}\|\|D_n - A\|\|\lambda(\lambda + A)^{-1}\| \\
&\leq \|D_n - A\|.
\end{aligned}$$

This implies that

$$\int_0^\infty \lambda D_n(\lambda + D_n)^{-1} d\mu(\lambda) \to \int_0^\infty \lambda A(\lambda + A)^{-1} d\mu(\lambda).$$

Consequently $f(D_n) \to f(A)$. Therefore if $A \in \mathbb{M}_n$ is accretive, then

$$f(A) = a + bA + \int_0^\infty \lambda A(\lambda + A)^{-1} d\mu(\lambda).$$

This completes the proof. □

Below, the authors presented a generalization of the result in Proposition 6.1 of Bedrani et al. [3].

Proposition 6.2. *[16, Proposition 2.2] Let $f : (0, \infty) \to \mathbb{R}$ be an operator monotone function and $A \in \mathbb{M}_n$ be an accretive matrix. Then for any unital positive linear map Φ on $\mathbb{M}_n$, the following inequalities hold*

$$\mathcal{R}f(\Phi(A)) \geq f(\mathcal{R}\Phi(A)) = f(\Phi(\mathcal{R}A)) \geq \Phi(f(\mathcal{R}A)). \tag{6.5}$$

In the following proposition, the authors established a reverse-type counterpart of Proposition 6.2.

Proposition 6.3. *[16, Proposition 2.3] Let $f : (0, \infty) \to \mathbb{R}$ be an operator monotone function and $A \in \mathcal{S}_\alpha$. Then for any unital positive linear map Φ on $\mathbb{M}_n$, the following inequality holds*

$$\mathcal{R}\Phi\left(f(A)\right) \leq \sec^2\alpha\ \mathcal{R}f\left(\Phi(A)\right).$$

Proof. We know that

$$\begin{aligned}
&\mathcal{R}(\lambda A(\lambda+A)^{-1}) = \mathcal{R}(\lambda^{-1}+A^{-1})^{-1} \\
&\leq (\lambda^{-1}+\mathcal{R}(A^{-1}))^{-1} && \text{(by Lemma 1.1)} \\
&\leq (\lambda^{-1}+\cos^2\alpha(\mathcal{R}A^{-1}))^{-1} = \sec^2\alpha((\lambda\cos^2\alpha)^{-1}+(\mathcal{R}A)^{-1})^{-1} \\
&= \lambda\mathcal{R}A(\lambda\cos^2\alpha+\mathcal{R}A)^{-1},
\end{aligned}$$

and

$$\begin{aligned}
\lambda\cos^2\alpha\mathcal{R}A(\lambda\cos^2\alpha+\mathcal{R}A)^{-1} &= ((\lambda\cos^2\alpha)^{-1}+(\mathcal{R}A)^{-1})^{-1} \\
&\leq (\lambda^{-1}+(\mathcal{R}A)^{-1})^{-1} = \lambda\mathcal{R}A(\lambda+\mathcal{R}A)^{-1}.
\end{aligned}$$

Now, we obtain

$$\begin{aligned}
\cos^2\alpha\mathcal{R}f(A) &= \cos^2\alpha\left(a+b\mathcal{R}A+\int_0^\infty \mathcal{R}\big(\lambda A(\lambda+A)^{-1}\big)d\mu(\lambda)\right) \\
&\leq a+b\mathcal{R}A+\cos^2\alpha\int_0^\infty \lambda\mathcal{R}A(\lambda\cos^2\alpha+\mathcal{R}A)^{-1}d\mu(\lambda) \\
&\leq a+b\mathcal{R}A+\int_0^\infty \lambda\mathcal{R}A(\lambda+\mathcal{R}A)^{-1}d\mu(\lambda) = f(\mathcal{R}A). \qquad (6.6)
\end{aligned}$$

From (6.5) and (6.6), we obtain

$$\cos^2\alpha\Phi(\mathcal{R}f(A)) \leq \Phi(f(\mathcal{R}A)) \leq \mathcal{R}f(\Phi A).$$

This completes the proof. □

In [30], the authors established that if $0 < m_1 I \leq A \leq M_1 I$, $0 < m_2 I \leq B \leq M_2 I$, and $0 < \nu \leq 1$, then

$$K(h,\nu)\,\Phi(A)\sharp_\nu\Phi(B) \leq \Phi(A\sharp_\nu B), \tag{6.7}$$

where $h = \dfrac{M_1 M_2}{m_1 m_2}$ and $K(h,\nu)$ denotes the generalized Kantorovich constant.

This section aims to present another complementary inequality to (6.9). Let A and B be positive definite operators on a Hilbert space $\mathcal{H}$, and let Φ be a unital positive linear map on $\mathcal{B}(\mathcal{H})$. For the weighted geometric mean, the following inequality holds:

$$\Phi(A\sharp_t B) \leq \Phi(A)\sharp_t\Phi(B), \tag{6.8}$$

where $t \in (0,1]$.

A complementary counterpart to inequality (6.8) is the following significant result from [30]: Let $0 < m_1 I \leq A \leq M_1 I$, $0 < m_2 I \leq B \leq M_2 I$, and $0 < t \leq 1$. Then

$$K(h,t)\,\Phi(A)\sharp_t\Phi(B) \leq \Phi(A\sharp_t B), \tag{6.9}$$

where $h = \dfrac{M_1 M_2}{m_1 m_2}$ and $K(h,t)$ is the generalized Kantorovich constant.

In the particular case where $t = 2$ or $t = -1$, one obtains $K(h,2) = K(h,-1) = \dfrac{(1+h)^2}{4h}$, which is the classical Kantorovich constant.

The generalized Kantorovich constant $K(h,t)$ satisfies the following properties:

$$K(h,t) = K(h,1-t),$$
$$0 < K(h,t) \leq 1 \quad \text{for all } 0 < t \leq 1,$$

and $K(h,t)$ is decreasing on $(0,\frac{1}{2}]$ and increasing on $(\frac{1}{2},1]$. Consequently, for all real t, the minimum value is attained at $t=\frac{1}{2}$, namely

$$K(h) = K\big(h,\tfrac{1}{2}\big) = \frac{2h^{1/4}}{1+h^{1/2}} \leq K(h,t).$$

Next, we recall Kadison's inequality and Choi's inequality:

$$\Phi(A^2) \geq \Phi(A)^2, \quad \Phi(A^{-1}) \geq \Phi(A)^{-1}, \tag{6.10}$$

which hold for any unital positive linear map Φ and any positive definite matrix A. along with their complementary counterparts, which hold under the assumption $0 < m \leq A \leq M$:

$$\frac{(m+M)^2}{4mM}\,\Phi(A)^2 \geq \Phi(A^2), \quad \frac{(m+M)^2}{4mM}\,\Phi(A)^{-1} \geq \Phi(A^{-1}). \tag{6.11}$$

The following inequality unifies the two inequalities in (6.10) into a single general form:

$$\Phi(BA^{-1}B) \geq \Phi(B)\,\Phi(A)^{-1}\,\Phi(B). \tag{6.12}$$

Moreover, when $0 < m \leq A, B \leq M$, the subsequent inequality serves as a complementary counterpart to (6.12):

$$\frac{(m+M)^2}{4mM}\,\Phi(B)\,\Phi(A)^{-1}\,\Phi(B) \geq \Phi(BA^{-1}B). \tag{6.13}$$

The following result, due to Ghazanfari et al. [16, Theorem 4.4], provides a two-sided refinement of Jensen's operator inequality for operator convex functions, incorporating the Kantorovich constant $K(h,2)$. It establishes sharp bounds for $\Phi(f(A))$ in terms of $f(\Phi(A))$ under the usual operator bounds $mI \leq A \leq MI$.

Theorem 6.4. *[16, Theorem 4.4] If f is an operator convex function on $(0,\infty)$, Φ is a unital positive linear map on $\mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$ with $0 < mI \leq A \leq MI$. Then*

$$K(h,2)f(\Phi(A)) \geq \Phi(f(A)) \geq f(\Phi(A)), \tag{6.14}$$

where $h = \frac{M}{m}$.

Proof. It is known that every operator convex function f on $(0,\infty)$ has a special integral representation as follows:

$$f(t) = \alpha + \beta t + \gamma t^2 + \int_0^\infty \frac{\lambda t^5}{\lambda + t} d\mu(\lambda), \tag{6.15}$$

where α, β are real numbers, $\gamma \geq 0$, and μ is a positive finite measure. Thus

$$\Phi(f(A)) = \alpha 1_K + \beta\Phi(A) + \gamma\Phi(A^2) + \int_0^\infty \Phi(\lambda A^2(\lambda + A)^{-1}) d\mu(\lambda),$$

and

$$f(\Phi(A)) = \alpha 1_K + \beta\Phi(A) + \gamma\Phi(A)^2 + \int_0^\infty \lambda\Phi(A)^2(\lambda + \Phi(A))^{-1} d\mu(\lambda).$$

By (6.11), we have

$$\begin{aligned}\Phi(\lambda A^2(\lambda + A)^{-1}) &= \lambda\Phi(A^2(\lambda + A)^{-1}) = \lambda\Phi(A(\lambda + A)^{-1}A)\\ &\geq \lambda\Phi(A)(\Phi(\lambda + A))^{-1}\Phi(A) = \lambda\Phi(A)(\lambda + \Phi(A))^{-1}\Phi(A)\\ &= \lambda\Phi(A)^2(\lambda + \Phi(A))^{-1}.\end{aligned}$$

Therefore $\Phi(f(A)) \geq f(\Phi(A))$, since $\Phi(A^2) \geq \Phi(A)^2$.
On the other hand, from (6.12) or (6.13), we get

$$\begin{aligned}&\Phi(\lambda A^2(\lambda + A)^{-1}) = \lambda\Phi(A^2(\lambda + A)^{-1}) = \lambda\Phi(A(\lambda + A)^{-1}A)\\ &\leq K(h,2)\lambda\Phi(A)(\Phi(\lambda + A))^{-1}\Phi(A) = K(h,2)\lambda\Phi(A)(\lambda + \Phi(A))^{-1}\Phi(A)\\ &= K(h,2)\lambda\Phi(A)^2(\lambda + \Phi(A))^{-1}.\end{aligned}$$

Therefore $\Phi(f(A)) \leq K(h,2)f(\Phi(A))$, since $\Phi(A^2) \leq K(h,2)\Phi(A)^2$. □

Remark on Pólya-type inequalities. Before concluding this review, we briefly mention Pólya-type inequalities and their extensions to sector matrices. Although this family of inequalities differs in nature from the operator mean inequalities discussed throughout the paper, we include it here for completeness. The classical Pólya inequality states that for any non-negative real numbers a and b,

$$\int_0^1 a^t b^{1-t}\, dt \leq \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right). \tag{6.16}$$

Zou [45] established a matrix analogue of (6.16) for all positive definite matrices $A, B \in \mathbb{M}_n$, namely

$$\int_0^1 A\sharp_t B\, dt \leq \frac{1}{3}\left(2A\sharp B + A\nabla B\right). \tag{6.17}$$

Furthermore, the same author proved the following Hilbert–Schmidt norm inequality of Pólya type:

$$\left\|\int_0^1 A^t X B^{1-t}\, dt\right\|_2 \leq \frac{1}{3}\left\|2A^{1/2}XB^{1/2} + \frac{AX + XB}{2}\right\|_2, \tag{6.18}$$

where $A, B, X \in \mathbb{M}_n$ and A, B are positive semidefinite.

Ghazanfari and Malekinejad [17] extended the classical Pólya inequality to the setting of sectorial matrices, as follows.

Theorem 6.5. *[17, Theorem 7] Let $A, B \in \mathcal{S}_\alpha$. Then, for $t \in [0,1]$,*

$$\left\|\int_0^1 (A\sharp_t B)\, dt\right\| \leq \sec^3\alpha \left\|\frac{2}{3}(A\sharp_t B) + \frac{1}{3}A\nabla_t B\right\|.$$

Proof. By Lemma 1 of [15], we have $(\int_0^1 (A\sharp_t B)dt)^* = \int_0^1 (A\sharp_t B)^* dt$.
Therefore $\mathcal{R}\int_0^1 (A\sharp_t B)dt = \int_0^1 \mathcal{R}(A\sharp_t B)dt$. By Theorem 3 of [36],

$$\begin{aligned} 0 \leq \int_0^1 (\mathcal{R}A\sharp_t\mathcal{R}B)dt \leq \mathcal{R}\int_0^1 (A\sharp_t B)dt &= \int_0^1 \mathcal{R}(A\sharp_t B)dt \\ &\leq \sec^2\alpha \int_0^1 (\mathcal{R}A\sharp_t\mathcal{R}B)dt \\ &\leq \sec^2\alpha \left(\frac{2}{3}\mathcal{R}A\sharp_t\mathcal{R}B + \frac{1}{3}\mathcal{R}A\nabla_t\mathcal{R}B\right) \\ &\leq \sec^2\alpha\mathcal{R}\left(\frac{2}{3}(A\sharp_t B) + \frac{1}{3}A\nabla_t B\right). \end{aligned} \tag{6.19}$$

Using Lemma2.3 and (6.19), we get

$$\begin{aligned} \left\|\int_0^1 (A\sharp_t B)dt\right\| &\leq \sec\alpha \left\|\mathcal{R}\int_0^1 (A\sharp_t B)dt\right\| \\ &\leq \sec^3\alpha \left\|\mathcal{R}\left(\frac{2}{3}(A\sharp_t B) + \frac{1}{3}A\nabla_t B\right)\right\| \\ &\leq \sec^3\alpha \left\|\frac{2}{3}(A\sharp_t B) + \frac{1}{3}A\nabla_t B\right\|. \end{aligned}$$

This completes the proof. □

7. Conclusion

This review examines the extension of operator relations from positive operators (positive definite matrices) to accretive operators (sectorial matrices). The principal finding is that inequalities valid for positive operators persist for accretive operators when scaled by a correction factor depending on the sectorial angle α. More strikingly, certain fundamental equalities such as specific trace or determinant identities that hold exactly in the positive setting are not preserved; instead, they transform into inequalities with a correction factor. This transformation has been rigorously established only for a limited subset of equalities, leaving a systematic characterization as an open problem. Theoretically, this reveals that the passage from positivity to accretivity induces a structural shift: exact identities become quantitatively perturbed. Practically, the correction factor provides a precise measure of deviation from positivity, which is valuable in the stability analysis of dynamical systems and the numerical treatment of non-self-adjoint problems. Key challenges remain: identifying which equalities undergo this transformation, determining optimal correction factors, and extending the framework to α-accretive or block-structured operators. Addressing these gaps will deepen the operator-theoretic understanding of sectorial structures and broaden their applicability in mathematical physics and control theory.

Acknowledgement

The author would like to thank the referees for valuable suggestions that have been implemented in the final version of the manuscript.

Declaration

The author declare that there is no conflict of interest regarding the publication of this paper.

Financial Support

The author received no specific funding for this work.

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